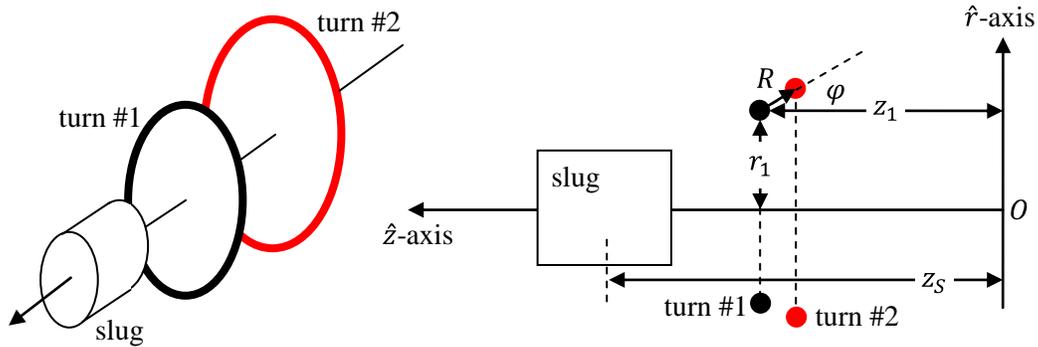


## An Axial Coil Gun

### Addendum 2 – The optimal shape for the coil

Reference is made to the paper titled *The physics of an axial coil gun*. The coils used in that paper were solenoids, in which the turns were wound side-by-side in cylinders. That may not be the best shape for a coil in a coil gun. The purpose of this paper is to explore whether there is a better shape.

We will approach this question as follows. We will start with simplest coil possible – a single circular loop of wire. We will then imagine adding a second loop carrying the same current. The question we will ask is: where is the best place to put that second loop? The configuration we will examine is shown in the following figure.



The  $\hat{r}$ - $\hat{z}$  co-ordinate frame of reference is defined as in the previous paper, with the  $\hat{z}$ -axis coincident with the central axis of the coil and pointing in the direction from which the slug approaches the coil. The  $\hat{r}$ -axis is any ray perpendicular to the  $\hat{z}$ -axis. In the previous paper, the “origin”, that is, the intersection between the  $\hat{z}$ - and  $\hat{r}$ -axes, was always placed at the center of the coil. In the case at hand, the starting coil is only one turn of wire, which is labeled “turn #1”. We do not know where the subsequent turns will be placed. For simplicity, therefore, the origin is shown as being fixed some arbitrary distance to the right of the first turn, so that the  $\hat{z}$ -co-ordinate of the starting turn is equal to  $z_1$ . The radius of the starting turn has been set to  $r_1$ . It is appropriate to think of turn #1 as the “leading” turn of the coil. As the slug is accelerated, it will encounter turn #1 first.

Now, we add a second turn, which is labeled “turn #2” and is highlighted in red. Rather than define this new turn by its absolute  $z$ - and  $r$ - co-ordinates, we will define its wire centerline relative to that of turn #1. To be precise, we will define a length  $R$  and an angle  $\varphi$  which describe where the top centerline of turn #2 is relative to the top centerline of turn #1.

Note first that the new turn must be located to the right of turn #1 in the figure above. If turn #2 was to the left of turn #1, then it would be the leading turn and the slug would encounter it first. The roles of the two turns would be reversed. We will require that the separation distance  $R$  be positive, so that the two turns cannot be coincident. Physically, distance  $R$  cannot be less than the diameter of the wire, including the insulation thickness of the wire from which the turns are constructed. Angle  $\varphi$  is positive if turn #2 has a bigger radius than turn #1. Because turn #2 must be further to the right than turn #1 (or, at least, not to the left of turn #1), angle  $\varphi$  must be in the range  $-90^\circ \leq \theta \leq +90^\circ$ .

I have made the implicit assumption that the two turns are coaxial. Their common axis passes through the geometric center of the slug, and all three are centered on the  $\hat{z}$ -axis. In the figure, the geometric center

of the slug is shown a distance  $z_5$  to the left of the origin. While we are making assumptions, let us also make the assumption that the same current flows through both turns.

Let me write down here, for ready reference later, the absolute radius  $r_2$  of turn #2 and the  $z$ -co-ordinate  $z_2$  of its center. In Cartesian co-ordinates:

$$\left. \begin{aligned} r_2 &= r_1 + R \sin \varphi \\ z_2 &= z_1 - R \cos \varphi \end{aligned} \right\} \quad (1)$$

The task is to find the combination of  $R$  and  $\varphi$ , which together define the size and placement of turn #2, which gives the slug the maximum amount of acceleration for a given flow of current.

This is too big a problem to solve analytically at one go. The magnetic field generated by the two turns varies throughout the volume of the slug. It also changes with time as the slug approaches the turns. Let us attack a simpler problem, and consider only a very small volume of the slug, which I will refer to as a ‘‘mote’’. If the physical size of the mote is small enough, we will be able to assume that the magnetic field intensity is constant throughout the mote’s volume. We will restrict the mote a little bit more, and require that it lie on the  $z$ -axis. We will use the symbol  $z_M$  for the distance of the mote from the origin. These two assumptions simplify the analysis significantly.

The force acting on the slug is, of course, the sum of the forces acting on all the little motes which make up its total volume. It is possible that changing the size and position of the second turn increases the force in some places inside the slug and decreases it at others. But, I am hoping that is not the case. I hope that changing the characteristics of the second turn affects all places inside the slug in some systematic way. If that is the case, then what we learn about the mote will apply in general to the entire slug.

### The components of the magnetic field at the mote’s location, due to turn #1 only

Let us begin. We can calculate the magnetic field which the current flowing in turn #1 generates at all locations in space, including the location of the mote. In another paper, titled *The force on a cylindrical steel slug inside a finite solenoid*, I set out expressions for the components of the magnetic field intensity generated by a solenoid, which is a collection of single turns placed face-to-face. The magnetic field generated by a solenoid is radially-symmetric and has two spatial components at every point in space: an axial component  $B_z$  pointing in the direction parallel to the  $\hat{z}$ -axis and a radial component  $B_r$  pointing directly outwards from the solenoid’s centerline. At any point of interest  $\vec{P}$ , whose positional co-ordinates are  $P_z$  and  $P_r$ , the two components of the magnetic field vector are:

$$\left. \begin{aligned} B_r &= \frac{\mu_0 I}{4\pi} \sum_{m=1}^{m=N_{layers}} \sum_{n=1}^{n=N_{turns}} \sum_{\theta=0}^{\theta=2\pi} \frac{r_{Mth\ layer} (P_z - z_{Nth\ turn}) \Delta\theta \cos \theta}{[P_r^2 + (P_z - z_{Nth\ turn})^2 + r_{Mth\ layer}^2 - 2P_r r_{Mth\ layer} \cos \theta]^{3/2}} \\ B_z &= \frac{\mu_0 I}{4\pi} \sum_{m=1}^{m=N_{layers}} \sum_{n=1}^{n=N_{turns}} \sum_{\theta=0}^{\theta=2\pi} \frac{-r_{Mth\ layer} (P_r \cos \theta - r_{Mth\ layer}) \Delta\theta}{[P_r^2 + (P_z - z_{Nth\ turn})^2 + r_{Mth\ layer}^2 - 2P_r r_{Mth\ layer} \cos \theta]^{3/2}} \end{aligned} \right\} \quad (1)$$

The outer summation is a summation over all of the layers in the winding, where  $r_{Mth\ layer}$  is the radius of the  $M^{th}$  layer in the winding from the solenoid’s centerline. The middle summation is a summation over all of the turns in each layer, where  $z_{Nth\ turn}$  is the axial displacement from the origin to the center of the plane which contains the  $N^{th}$  turn in each layer. The inner summation is a summation around the

circumference of the  $N^{\text{th}}$  turn in the  $M^{\text{th}}$  layer, where progress around the circumference is measured by angle  $\theta$ . In all instances, distances are measured to the centerline of the current-carrying wire.

If the solenoid consists of a single turn of wire, as does our turn #1, then the two outer summations are not relevant. We can simply set the radius  $r_{M^{\text{th}} \text{ layer}}$  to the radius of the turn #1 ( $r_1$ ) and the displacement  $z_{N^{\text{th}} \text{ turn}}$  to the axial displacement of turn #1 ( $z_1$ ). Then, Equation (1) reduces to:

$$\left. \begin{aligned} B_r &= \frac{\mu_0 I}{4\pi} \sum_{\theta=0}^{\theta=2\pi} \frac{r_1 (P_z - z_1) \Delta\theta \cos \theta}{[P_r^2 + (P_z - z_1)^2 + r_1^2 - 2P_r r_1 \cos \theta]^{3/2}} \\ B_z &= \frac{\mu_0 I}{4\pi} \sum_{\theta=0}^{\theta=2\pi} \frac{-r_1 (P_r \cos \theta - r_1) \Delta\theta}{[P_r^2 + (P_z - z_1)^2 + r_1^2 - 2P_r r_1 \cos \theta]^{3/2}} \end{aligned} \right\} \quad (2)$$

It gets even simpler. The point  $\vec{P}$  of interest to us is the center of the mote. Since the mote lies on the  $\hat{z}$ -axis, we can set  $P_r$  to zero. We defined the mote's axial distance from the origin in such a way that  $P_z$  is the same as  $z_M$ . With these changes, Equation (2) simplifies to:

$$\left. \begin{aligned} B_r &= \frac{\mu_0 I}{4\pi} \sum_{\theta=0}^{\theta=2\pi} \frac{r_1 (z_M - z_1) \Delta\theta \cos \theta}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \\ B_z &= \frac{\mu_0 I}{4\pi} \sum_{\theta=0}^{\theta=2\pi} \frac{r_1^2 \Delta\theta}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \end{aligned} \right\} \quad (3)$$

Of, yes, one other thing. Equations (1) through (3) assume that point  $\vec{P}$  lies in air, or in free space. If point  $\vec{P}$  lies inside a magnetic material, such as the mote, then these expressions for  $B_r$  and  $B_z$  must be multiplied by the relative permeability  $\mu_r$  of the material. We will make this change below once we get the expressions for the force. For now, we will continue to use  $B_r$  and  $B_z$  without the relative permeability, and continue to look at the magnetic field generated by the turn without the mote present.

The remaining summations are sums around a circle. The differentials  $\Delta\theta$  are small angles which, taken together, represent a complete circular traverse around turn #1. In the limit, as we make the differentials  $\Delta\theta$  smaller and smaller, the summations approach the following integrals:

$$\left. \begin{aligned} B_r &= \frac{\mu_0 I}{4\pi} \int_{\theta=0}^{\theta=2\pi} \frac{r_1 (z_M - z_1) \cos \theta}{[(z_M - z_1)^2 + r_1^2]^{3/2}} d\theta \\ B_z &= \frac{\mu_0 I}{4\pi} \int_{\theta=0}^{\theta=2\pi} \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} d\theta \end{aligned} \right\} \quad (4)$$

From the point-of-view of the integration variable  $\theta$ , the factors involving the radius  $r_1$  and the axial displacements  $z_1$  and  $z_M$  are constants, and can be removed from the integrals. We get:

$$B_r = \frac{\mu_0 I}{4\pi} \frac{r_1 (z_M - z_1)}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \int_{\theta=0}^{\theta=2\pi} \cos \theta d\theta \quad (5A)$$

and

$$B_z = \frac{\mu_0 I}{4\pi} \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \int_{\theta=0}^{\theta=2\pi} d\theta \quad (5B)$$

These two integrals can be integrated by inspection. The integral of cosine around a circle is zero. Physically, this means that the radial component of the magnetic field is zero, as one expects. The other integral sums to  $2\pi$  and gives the following result for the components of the magnetic field:

$$\left. \begin{aligned} B_r &= 0 \\ B_z &= \frac{\mu_0 I}{2} \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \end{aligned} \right\} \quad (6)$$

The axial component  $B_z$  consists of squares and the sum of squares, so it is always algebraically positive. Physically, this means that  $B_z$  points in the direction of the positive  $\hat{z}$ -axis – to the left in the figure above – at all points along the  $\hat{z}$ -axis.

Once we have fixed the size and position of turn #1, the only independent variable on the right-hand side of  $B_z$  is the axial displacement  $z_M$  of the mote. In the analysis below, we are going to need to know how quickly the magnetic field strength  $B_z$  changes as we change the location  $z_M$ . This is the quantity measured by the derivative of  $B_z$  with respect to  $z_M$ , which we can compute as follows:

$$\begin{aligned} \frac{dB_z}{dz_M} &= \frac{\mu_0 I}{2} \frac{d}{dz_M} \left\{ \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \right\} \\ &= \frac{\mu_0 I}{2} \left( -\frac{3}{2} \right) \left\{ \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{5/2}} \right\} \frac{d}{dz_M} [(z_M - z_1)^2 + r_1^2] \\ &= -\frac{3\mu_0 I}{2} \frac{(z_M - z_1)r_1^2}{[(z_M - z_1)^2 + r_1^2]^{5/2}} \end{aligned} \quad (7)$$

So long as the mote lies to the left of the center of turn #1, that is, that  $z_M > z_1$ , this derivative will be algebraically negative. Physically, this means that the strength of the magnetic field  $B_z$  decreases as the mote is moved further away from turn #1, and  $z_M$  increases. This also, was to be expected.

### The force acting on the mote, due to turn #1 only

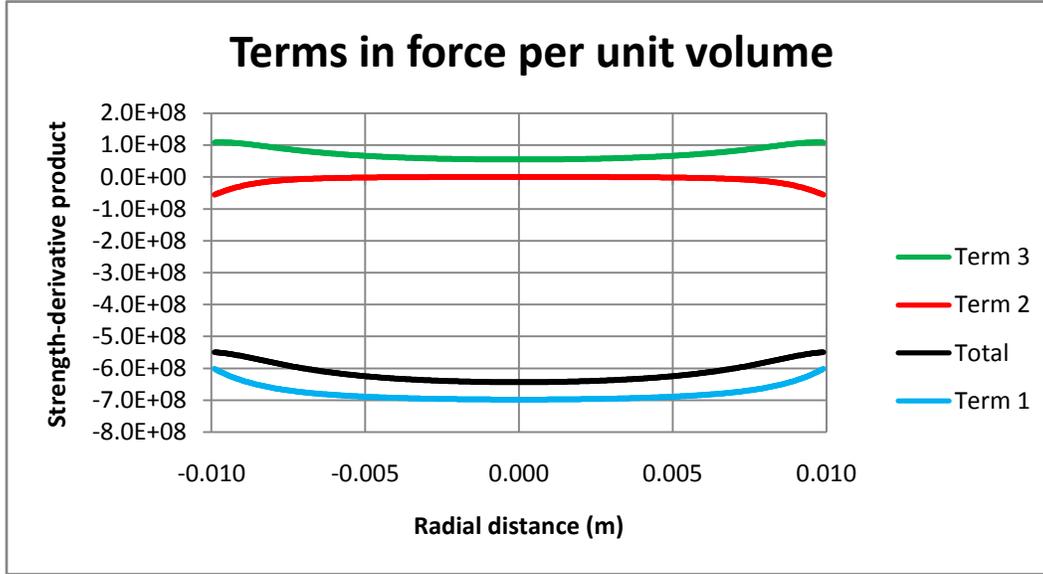
Now, let us now consider the force which turn #1 exerts on the mote. In the earlier paper titled *The physics of an axial coil gun*, we wrote down in Equation (5) therein the expression for the force per unit volume at any point  $\vec{P}$  inside the slug. That expression was:

$$\vec{f}(P_r, P_z) = \frac{\mu_r^2}{\mu_0} \left( B_r \frac{\partial B_r}{\partial r} + B_r \frac{\partial B_z}{\partial z} + B_z \frac{\partial B_r}{\partial z} \right) \hat{r} + \frac{\mu_r^2}{\mu_0} \left( B_z \frac{\partial B_z}{\partial z} + B_r \frac{\partial B_z}{\partial r} + B_z \frac{\partial B_r}{\partial r} \right) \hat{z} \quad (8)$$

where  $\vec{f}(P_r, P_z)$  the force per unit volume at point  $(P_r, P_z)$ . The force has two components:  $f_r$  in the radial direction and  $f_z$  in the axial direction. We found in the previous paper that all three terms in the radial component  $f_r$  are zero for any physical configuration which is radially-symmetric. On the other hand, the three terms in the axial component  $f_z$  are non-zero. (Incidentally, the relative permeability  $\mu_r$  has now

been added in Equation (8) to reflect the fact that point  $\vec{P}$  is inside the mote. One of the  $\mu_r$ 's arises from the magnetic field intensity and the other arises from the spatial derivatives of the magnetic field intensity.)

The three terms in the axial component  $f_z$  do not make equal contributions to the net force on the slug or mote. We found in the earlier paper that the first term  $-B_z \partial B_z / \partial z$  dominates the other two terms. As an example, I have repeated here a graph from the earlier paper which shows how the three terms vary across a diameter of the face of a solenoid. The net force per unit volume is negative, acting in the direction of the negative  $\hat{z}$ -axis and so attracting the slug or mote towards the center of the solenoid.



The three terms have different magnitudes. Term 2 is very small. Term 3 is slightly repulsive. Term 1, which is the blue curve in the figure, is by far the dominant term. At the center of the solenoid, it is about 14 times larger than Term 3. The dominance of the first term should hold for a single turn of wire, which is just a degenerate case of a solenoid. If we use the first term only, then we can approximate the force per unit volume acting on the mote as:

$$\vec{f}(P_r, P_z) \approx \frac{\mu_r^2}{\mu_0} \left( B_z \frac{\partial B_z}{\partial z} \right) \hat{z} \quad (9)$$

We already have an expression for  $B_z$  at the mote [Equation (6)] and an expression for the derivative of the axial component of the magnetic field with respect to the axial displacement [Equation (7)]. Substituting them into Equation (9) allows us to write the force per unit volume of the mote as:

$$\begin{aligned} f_z(z_M)|_{turn \#1} &\approx \frac{\mu_r^2}{\mu_0} \left\{ \frac{\mu_0 I}{2} \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} \right\} \left\{ -\frac{3\mu_0 I}{2} \frac{(z_M - z_1)r_1^2}{[(z_M - z_1)^2 + r_1^2]^{5/2}} \right\} \\ &= -\frac{3\mu_r^2 \mu_0 I^2}{4} \frac{(z_M - z_1)r_1^4}{[(z_M - z_1)^2 + r_1^2]^4} \end{aligned} \quad (10)$$

This is the force per unit volume (approximately) which turn #1 exerts on a mote of magnetic material located on the  $\hat{z}$ -axis a distance  $z_M$  from the origin.

### The energy imparted to the mote by turn #1 only

If the mote has a volume given by  $V_M$ , then the total force (approximately) which turn #1 exerts on the mote when it is at displacement  $z_M$  will be equal to the force per unit volume multiplied by the volume, thus:

$$F_z(z_M)|_{turn \#1} \approx -\frac{3\mu_r^2\mu_0 I^2 V_M}{4} \frac{(z_M - z_1)r_1^4}{[(z_M - z_1)^2 + r_1^2]^4} \quad (11)$$

If the mote is allowed to accelerate under the influence of this force, from a distance which is very far away (that is,  $z_M \rightarrow \infty$ ) up to the center of turn #1, then the work done on the mote will be equal to the cumulative sum of the force multiplied by the distance through which it acts, that is:

$$\begin{aligned} W|_{turn \#1} &= \int_{z_M=\infty}^{z_M=z_1} F_z(z_M)|_{turn \#1} dz_M \\ &\approx -\frac{3\mu_r^2\mu_0 I^2 V_M r_1^4}{4} \int_{z_M=\infty}^{z_M=z_1} \frac{(z_M - z_1)}{[(z_M - z_1)^2 + r_1^2]^4} dz_M \quad (12) \end{aligned}$$

This integral can be taken in closed form, as follows:

$$\begin{aligned} \int \frac{x dx}{(x^2 + a^2)^4} &= \frac{1}{2} \int \frac{d(x^2 + a^2)}{(x^2 + a^2)^4} \\ &= -\frac{1}{6(x^2 + a^2)^3} + C \end{aligned}$$

so that:

$$\begin{aligned} W|_{turn \#1} &\approx \left(-\frac{3\mu_r^2\mu_0 I^2 V_M r_1^4}{4}\right) \left(-\frac{1}{6}\right) \left\{ \frac{1}{[(z_M - z_1)^2 + r_1^2]^3} \Big|_{z_M=\infty}^{z_M=z_1} \right\} \\ &= \frac{\mu_r^2\mu_0 I^2 V_M r_1^4}{8} \left\{ \frac{1}{[0 + r_1^2]^3} - 0 \right\} \\ &= \frac{\mu_r^2\mu_0 I^2 V_M}{8r_1^2} \quad (13) \end{aligned}$$

The kinetic energy of the mote will be equal to this amount of work done by turn #1. The kinetic energy imparted to the mote is inversely proportional to the radius of turn #1. In order to maximize the final speed of the mote, turn #1 should have as small a radius as possible. (Of course, in our application, the mote must be able to pass through turn #1.)

We ended the above integral when the mote reached the center of the coil, where  $z_M = z_1$ . Going further makes no sense. Once the mote lies to the right of the center of turn #1, then  $z_M - z_1 < 0$ , the force per unit volume becomes algebraically positive (decelerating the mote) and the work done on the mote starts to decrease.

### The analysis with two turns

Now, let us repeat the analysis assuming that there is a second turn present. For the time being, I will use Cartesian co-ordinates for turn #2, that is,  $r_2$  and  $z_2$ , which correspond to the dimensions  $r_1$  and  $z_1$  for turn #1. The magnetic fields generated by the two turns separately can be added together at any point in space. Vector addition would normally be used but, since the magnetic field is entirely axial at all points along the  $\hat{z}$ -axis, scalar addition of the axial components will suffice. We get:

$$\begin{aligned} B_z|_{two\ turns} &= \frac{\mu_0 I}{2} \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} + \frac{\mu_0 I}{2} \frac{r_2^2}{[(z_M - z_2)^2 + r_2^2]^{3/2}} \\ &= \frac{\mu_0 I}{2} \left\{ \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} + \frac{r_2^2}{[(z_M - z_2)^2 + r_2^2]^{3/2}} \right\} \end{aligned} \quad (14)$$

Taking the derivative, as before, gives:

$$\left. \frac{dB_z}{dz_M} \right|_{two\ turns} = -\frac{3\mu_0 I}{2} \left\{ \frac{(z_M - z_1)r_1^2}{[(z_M - z_1)^2 + r_1^2]^{5/2}} + \frac{(z_M - z_2)r_2^2}{[(z_M - z_2)^2 + r_2^2]^{5/2}} \right\} \quad (15)$$

Then, the force per unit volume acting on the mote when both turns are carrying the same current can be found by substituting Equations (11) and (12) into Equation (9), with the result:

$$f_z(z_M)|_{two\ turns} \approx -\frac{3\mu_r^2\mu_0 I^2}{4} \left\{ \begin{aligned} &\left( \frac{r_1^2}{[(z_M - z_1)^2 + r_1^2]^{3/2}} + \dots \right) \left( \frac{(z_M - z_1)r_1^2}{[(z_M - z_1)^2 + r_1^2]^{5/2}} + \dots \right) \\ &\dots + \frac{r_2^2}{[(z_M - z_2)^2 + r_2^2]^{3/2}} \left( \dots + \frac{(z_M - z_2)r_2^2}{[(z_M - z_2)^2 + r_2^2]^{5/2}} \right) \end{aligned} \right\} \quad (16)$$

We can group terms to write this as:

$$f_z(z_M)|_{two\ turns} \approx -\frac{3\mu_r^2\mu_0 I^2}{4} \left\{ \begin{aligned} &\frac{(z_M - z_1)r_1^4}{[(z_M - z_1)^2 + r_1^2]^4} + \dots \\ &\dots + \frac{(z_M - z_2)r_1^2 r_2^2}{[(z_M - z_1)^2 + r_1^2]^{3/2} [(z_M - z_2)^2 + r_2^2]^{5/2}} + \dots \\ &\dots + \frac{(z_M - z_1)r_1^2 r_2^2}{[(z_M - z_2)^2 + r_2^2]^{3/2} [(z_M - z_1)^2 + r_1^2]^{5/2}} + \dots \\ &\dots + \frac{(z_M - z_2)r_2^4}{[(z_M - z_2)^2 + r_2^2]^4} \end{aligned} \right\} \quad (17)$$

There are four terms in the curly brackets. The one on the first line is exactly the same as the force per unit volume generated by turn #1 alone, as set out in Equation (11). The term on the last line has the same form and can be thought of as the force per unit volume generated by turn #2 in isolation. The middle two terms are symmetric in the subscripts "1" and "2" and can be thought of as the additional force generated by turn #1 when turn #2 is present and by turn #2 when turn #1 is present. In a sense, the first term corresponds to the self-inductance of turn #1, the last term corresponds to the self-inductance of turn #2 and the middle two terms correspond to their mutual inductance.

Consider what happens when we make the two turns coincide, which we can do by setting  $r_2 = r_1$  and  $z_2 = z_1$ . All four terms become identical and the force  $f_z(z_M)|_{two\ turns}$  becomes equal to four times the force generated by turn #1 alone. That is exactly what happens if we double the current flowing through turn #1 alone:  $(2I)^2 = 4I^2$ .

Now, let us return to our task. We can multiply the force per unit volume in Equation (17) by the volume of the mote to find the total force acting on the mote, and then integrate this force over distance  $z_M$ , starting with the mote being far away ( $z_M \rightarrow \infty$ ). Where we should finish is not so clear. When we had a single turn, it was clear that things should stop when the mote got to the center of the turn. Any further, and the mote would start to decelerate. With two turns, it may or may not be advisable for the mote to pass through the center of the leading turn before the current is “turned off”. Turn #2, which lies closer to the origin than turn #1, could continue to accelerate the mote. Certainly, we will want to stop at or before the time the mote reaches the center of turn #2. How much before is an unanswered question.

Therefore, let us allow for two cases. In the first case, called Case A, we will stop the integration when the mote arrives at the center of the leading turn, turn #1. In Case B, we will allow the integration to continue until the time when the force per unit volume switches from negative to positive. A positive force per unit volume points in the direction of the positive  $\hat{z}$ -axis and will start to decelerate the mote. In symbolic terms, the two cases can be described as follows.

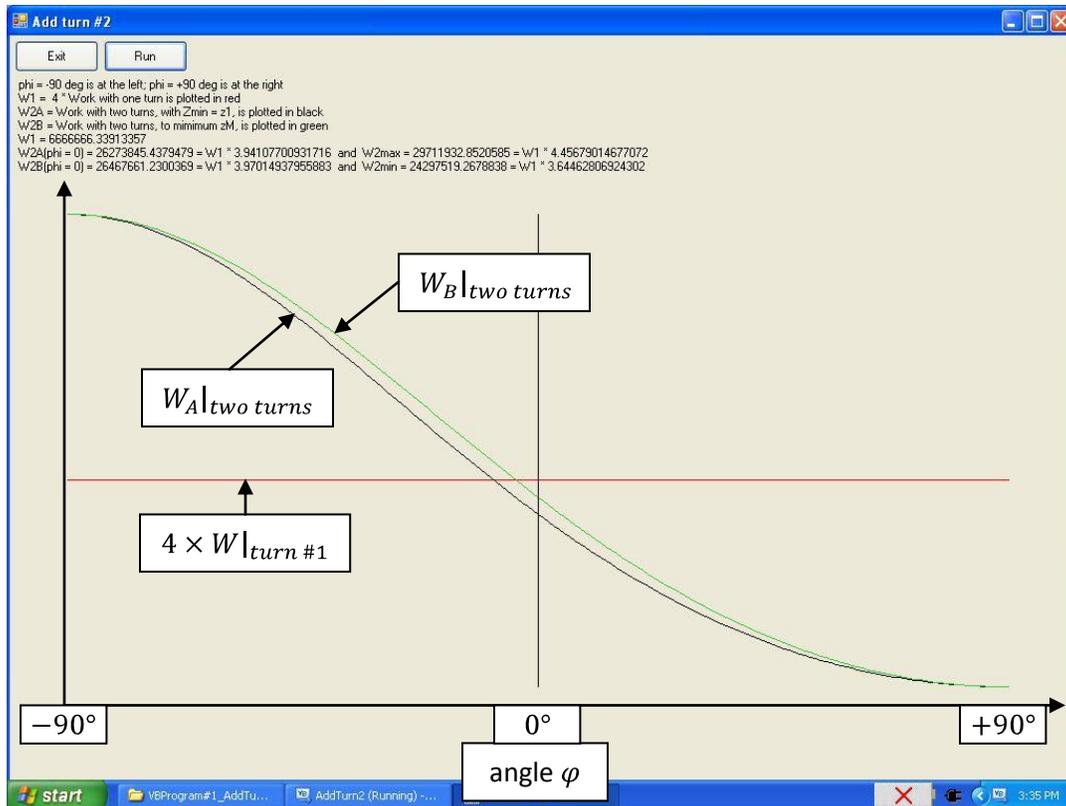
$$\left. \begin{array}{l} \text{Case A: } W_A|_{two\ turns} = -\frac{3\mu_r^2\mu_0 I^2 V_M}{4} \int_{z_M=\infty}^{z_M=z_1} [\text{Equation (17)}] dz_M \\ \text{Case B: } W_B|_{two\ turns} = -\frac{3\mu_r^2\mu_0 I^2 V_M}{4} \int_{z_M=\infty}^{z_M=?} [\text{Equation (17)}] dz_M \end{array} \right\} \quad (18)$$

The only difference between the two cases is where / when during the run we stop integration.

The integrals of the four terms in Equation (17) cannot be taken in closed form. Fortunately, the number of variables is small and all of them are distances. Resorting to a numerical integration should not disguise the important results. Let us suppose that turn #1 has a radius of 5 cm and that the separation distance between the centerlines of turn #1 and turn #2 is fixed at 5 mm. We will integrate both cases for 181 different values of angle  $\varphi$  between  $-90^\circ$  and  $+90^\circ$ .

The results are shown in the following plot, which is a screen-shot produced by the program which carried out the integration. The horizontal axis in the plot is the angle  $\varphi$  which locates the centerline of turn #2 around the centerline of turn #1.  $\varphi = -90^\circ$  defines the configuration in which turn #2 lies wholly inside turn #1, and is at the left.  $\varphi = +90^\circ$  defines the configuration in which turn #2 exactly surrounds turn #1, and is at the right.  $\varphi = 0^\circ$  defines the configuration in which turn #2 has the same radius as turn #1 and follows turn #2, just like the first two turns in a classical solenoid.

The vertical axis shows the work done on the mote, which equals its kinetic energy at the end of the “run”. I have not shown any units for the vertical axis since the numerical routine made no effort to account for the variables  $\mu_0$ ,  $\mu_r$ ,  $I$  and  $V_M$ , which were considered to be kept the same for all runs.



The red line shows the work done by turn #1 alone, multiplied by four. When there is only one turn, angle  $\varphi$  is not relevant. Hence, this line is horizontal.

The black line shows the work done on the mote in Case A, where the integration is stopped as soon as the mote reaches the center of the leading turn, turn #1. By travelling further under the influence of the force, so long as it is attractive, the mote could get a little more energy. This is Case B, which is shown by the green line.

Since our objective is to maximize the speed of the mote, the “best” value in the plot occurs when angle  $\varphi$  is  $-90^\circ$ , at the extreme left. In this configuration, the top centerline of turn #2 is directly below the top centerline of turn #1. The two turns lie in the same plane, but turn #2 has a smaller radius than turn #1. On the other hand, the “worst” value of angle  $\varphi$  is  $+90^\circ$ , at the extreme right of the plot. In this configuration, the top centerline of turn #2 is directly above the top centerline of turn #1. The two turns lie in the same plane, as in the “best” case, but this time turn #2 has a bigger radius than turn #1 and surrounds turn #1.

What these results suggest is that the radii of the turns are the most important factor determining the work done and that a smaller radius is better than a bigger radius. We came to both of these conclusions when we looked at the single turn in the preceding sections.

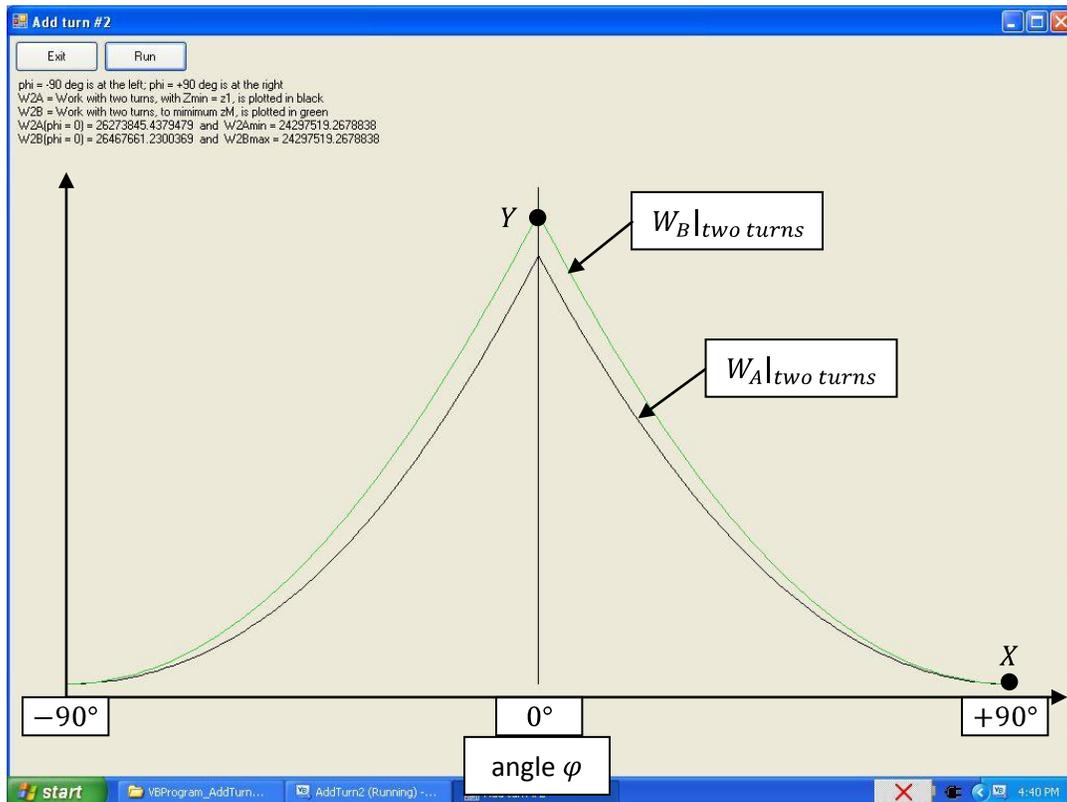
We can increase the work done by decreasing the radii of both turns. But, there is a limit to doing this. Notwithstanding everything else, the coil for a coil gun must have a inner diameter which is big enough to accommodate both the slug (as opposed to the mote) as well as whatever guiderails hold the slug on course during its run. If we set the radius of turn #1 to this minimum radius, then turn #2 cannot be permitted to have a smaller radius. If we set the radius of turn #2 to this minimum radius, then turn #1

cannot be permitted to have a smaller radius. In a moment or two, I will show the result from imposing this restriction.

Before doing that, I want to point out a second observation which can be made from the plot. It is not horizontally symmetric. Consider the work done when angle  $\varphi = 0^\circ$ , which defines the configuration where turn #2 is exactly the same size as turn #1 and is placed face-to-face with turn #1. The work done on the mote in both Case A and Case B is slightly less than four times the work done on the mote by turn #1 alone. In other words, two separate turns are never as powerful as one turn carrying twice the current.

The following screen-shot shows the result of repeating the analysis with the added restriction of a minimum acceptable diameter. Since turn #1 had a 5 cm radius in the numerical example above, I chose to set the minimum radius to 5 cm. At all times, the radii of both turns must equal or exceed 5 cm. For angles of  $\varphi$  between  $0^\circ$  and  $+90^\circ$ , the radius of turn #2 will be greater than the radius of turn #1. For angles between  $-90^\circ$  and  $0^\circ$ , the radius of turn #2 will be smaller. The following results were obtained by setting the radius of the smaller turn, either turn #1 or turn #2, to the minimum value and letting the radius of the larger turn be determined by angle  $\varphi$ .

There is no red curve in this plot. In the former plot, the red curve was four times the work done by turn #1 alone. In the new analysis, the radius of turn #1 is not the same in all the runs, so there is no single reference value for the work by turn #1. As before, though, the black curve is the energy received by the mote up to the time it reaches the center of the leading turn, turn #1. The green curve then shows how much more energy the mote receives if we let the current flow up until the instant when the force turns repulsive.



The work curves are now symmetric around the angle  $\varphi = 0^\circ$ . The “best” turn #2 is a second turn having the same radius as turn #1 (so that both radii are equal to the minimum radius allowed) and located right “behind” turn #1 from the point-of-view of the mote.

As before, it seems to be the case that minimizing the radius of turn #2 is the best strategy.

It would be wrong for the reader to conclude that adding turn #2 outside of turn #1 is a bad thing. That is definitely not true. All that can really be said is that adding turn #2 outside of turn #1 does not increase the work done on the mote as much as adding turn #2 behind turn #1. If have labeled point X in the plot above for the configuration where turn #2 exactly surrounds turn #1. In this configuration, the final kinetic energy of the mote is (a relative value excluding the variables  $\mu_0$ ,  $\mu_r$ ,  $I$  and  $V_M$  of) 24,297,514. Point Y in the plot labels the configuration where turn #2 is the same size as turn #1, and is placed side-by-side with turn #1. In this configuration, the final kinetic energy of the mote is (a relative value of) 26,467,661. Both of these values should be compared to the work done by turn #1 alone, which is found in the previous plot and has (a relative) value of 6,666,666. The three configurations are compared in the following table:

Configuration	Work	Percentage
4 times turn #1 alone	26,666,664	100.00%
turn #2 behind	26,467,661	99.25%
turn #2 outside	24,297,514	91.16%

If we need to add a second turn, the ideal place to put turn #2 is right on top of, or coincident with, turn #1. That is the first line of the table, in which the kinetic energy imparted to the mote is four times that of turn #1 by itself.

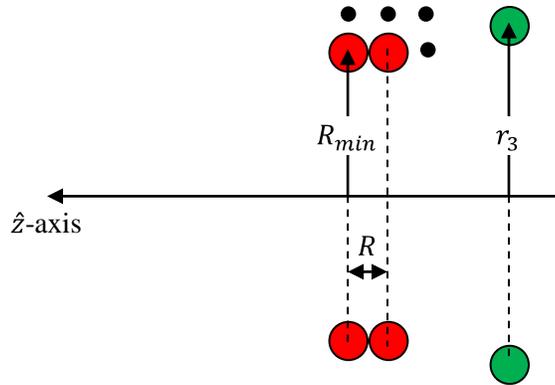
If we cannot make the two turns coincident, then the next best thing to do is to make turn #2 the same size as turn #1 and to put it right behind turn #1. That is the second line of the table. That gives the mote 99.25% as much kinetic energy as the ideal amount. (Note that 99.25% depends on the parameters chosen for the numerical integration, with the numbers here based on a minimum radius of 5 cm and a minimum centerline separation of 5 mm.)

An alternative is to make turn #2 larger than turn #1 by the minimum amount possible, and then to place turn #2 so the two turns lie in the same plane. That configuration is the third line of the table. It gives the mote 91.16% of the ideal amount. While this is not as much as the “behind” configuration in the second line, it is still far more than the kinetic energy from turn #1 alone.

### The analysis for subsequent turns

Now, let us add a third turn to the coil. The third turn will be characterized by the Cartesian co-ordinates  $r_3$  and  $z_3$ . What size and shape will make this third turn most effective in adding kinetic energy to the mote, assuming that this new turn carries the same current as the first two turns? The following figure shows a cross-section of the three-turn coil.

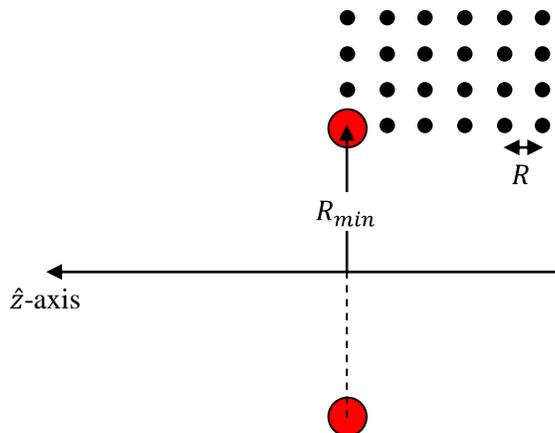
The cross-sections of the first two turns are shown in red. They have the same radius  $R_{min}$ , being the minimum allowed, and their planes are separated by distance  $R$ . I have shown the distance  $R$  as being the diameter of the wire itself, since this will almost always be the cause of the minimum separation. The new turn, turn #3, is shown in green.



Rather than consider all possible locations in which turn #3 would touch the first two turns, will we consider only four locations, shown as a grid of black dots. We will place turn #3 either directly outside of the first two turns or directly behind turn #2.

The alert reader may ask: why even consider placing turn #3 outside of the first two turns? We have seen that minimizing the radius maximizes the work. The answer is this. All else being the same, minimizing the radius is helpful, but we have only confirmed that for two turns. If we try to minimize the radius of the third turn, to give it the same radius as the other two, then we must place turn #3 behind turn #2, where it will be farther from the mote at all times than the other two turns. It could be that the increased displacement from the mote is more of a disadvantage than the increased radius of placing turn #3 outside of turn #1. At some point, the trade-off between radius and axial displacement from the mote should become the deciding issue. We may find that it is not the third turn, but some later turn, at which this occurs, but the axial displacement should eventually become the deciding factor.

We are not going to stop at three turns. We are going to build the coil from an arbitrary number of turns, which we will add to the coil in sequence, always placing the next turn in whatever location maximizes the kinetic energy imparted to the mote. The following figure shows the various positions we will allow for the turns.



Turn #1, shown in red, will always be placed in the lower-left corner of a square grid, shown by the black dots in the figure. Turn #1 will always be the leading turn, and will have the smallest radius possible. When we add turn #2, or any subsequent turn, its top centerline must coincide with one of the grid points. The points in the grid are separated by the minimum separation  $R$  in both the axial and radial directions.

For the numerical integration, I am now going to change the basic dimensions of the coil and wire so that they are closer to the dimensions used in the earlier paper. I will assume that the inside diameter of the coil, when finished, must be 1 cm. I will assume that the wire is #20 gauge enameled copper wire, which

has a radius of 0.812 mm. The radius to the centerline of turn #1 will therefore be equal to 1.0406 cm and the spacing between grid points will be 0.0812 cm.

To carry out the numerical integration, we must also generalize the expression for the force per unit volume set out in Equation (17). Equation (17) accommodates only two turns. It contains four terms, since it is the product of the magnetic field intensity (one term from each of two turns) and the derivative of the magnetic field intensity (also one term from each of two turns). If the coil consists of  $N$  turns, the force per unit volume will contain  $N^2$  terms. It is not worthwhile expanding this into closed form or even as a summation. Instead, we will code the numerical routine so that it handles a growing coil in a more systematic way.

Consider the configuration when then coil consists of  $N$  turns. We will know the radii  $r_i$  and the axial displacement from the origin  $z_i$  of each turn. To calculate the force per unit volume acting on the mote when it is at axial displacement  $z_M$ , one starts by calculating the total magnetic field intensity generated by the coil at  $z = z_M$ , by adding up the contributions to  $B_z$  made by each of the  $N$  turns, as follows:

$$\begin{aligned} B_z(z = z_M)|_{N \text{ turns}} &= \sum_{i=1}^N B_z|_{\text{turn } \#i} \\ &= -\frac{3\mu_r^2\mu_0 I^2}{4} \sum_{i=1}^N \frac{(z_M - z_i)r_i^4}{[(z_M - z_i)^2 + r_i^2]^4} \quad (19) \end{aligned}$$

Rather than deal with any closed form for the derivative, we will let the program approximate the derivative using the ratio of differences. We will calculate the total magnetic field intensity at an axial displacement slightly to the left of  $z = z_M$ , say, at  $z = z_M + \Delta z$ . Then, the derivative can be approximated by the following ratio of differences. The approximation improves, of course, as we make  $\Delta z$  smaller and smaller.

$$\left. \frac{dB_z(z = z_M)}{dz} \right|_{N \text{ turns}} \approx \frac{B_z(z = z_M + \Delta z)|_{N \text{ turns}} - B_z(z = z_M)|_{N \text{ turns}}}{\Delta z} \quad (20)$$

Now that we have the total magnetic field intensity and the derivative, we can calculate the force per unit volume at  $z = z_M$  by multiplying them together, as per Equation (9) to get:

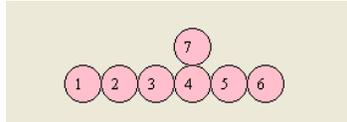
$$\begin{aligned} f_z(z_M)|_{N \text{ turns}} &\approx \frac{\mu_r^2}{\mu_0} \left( B_z \frac{\partial B_z}{\partial z} \right) \\ &= -\frac{\mu_r^2}{\mu_0} B_z(z = z_M)|_{N \text{ turns}} \left[ \frac{B_z(z = z_M + \Delta z)|_{N \text{ turns}} - B_z(z = z_M)|_{N \text{ turns}}}{\Delta z} \right] \quad (21) \end{aligned}$$

We can multiply this by the volume of the mote to calculate the total force acting on the mote  $F_z(z_M)|_{N \text{ turns}}$  and can have the computer program add up the products of the force and distance from a starting displacement far from the coil to the axial displacement at which the force per unit volume changes from negative (accelerating the mote) to positive (decelerating the mote)

The computer program which does this is listed in Appendix #A attached hereto. The program does not calculate the constant factors involving the permeabilities  $\mu_0$  and  $\mu_r$ , the current  $I$  and the volume  $V_M$  of the mote. Whatever these values are, they affect all runs by the same proportion and do not affect any comparison of the runs.

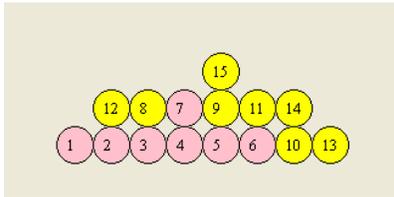
When the program has placed  $N$  turns, and prepares to place the  $N + 1^{\text{st}}$  turn, it proceeds as follows. It determines which locations in the grid are not already occupied by the first  $N$  turns. It tries all unoccupied locations. It places the  $N + 1^{\text{st}}$  turn in each unoccupied location in turn and calculates the work done on the mote during a run. It compares the work done on the mote with the  $N + 1^{\text{st}}$  turn in the various unoccupied locations and it places the turn in the location which imparts the most energy to the mote. Then, it proceeds to test the  $N + 2^{\text{nd}}$  turn.

As it turns out, the best size for turn #3 is the minimum radius and the best location for it is immediately following turn #2. The same holds true for turns 4 through 6. The first six turns describe a short single-layer solenoid. The interesting thing happens at the seventh turn. The following diagram shows the best size and location for turn #7.



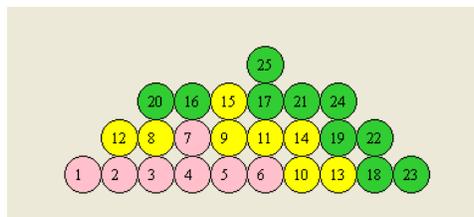
The seventh turn starts a second layer. Note that the seventh turn does not surround turn #1, where it would be “closer to the mote”. It surrounds turn #4, which is near the midpoint of the solenoid formed by the first six turns, in which location it can help accelerate the mote through the leading half of the solenoid.

Subsequent turns, the eighth and so on, are added in either the first layer or the second layer, taking advantage once again of the benefits of smaller radii. The next interesting thing happens at turn # 15. The following diagram shows the best size and location for turn #15.



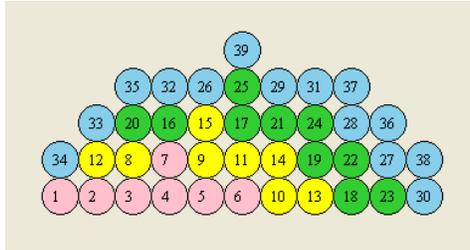
I have shown the wires in this second batch, which ends with the first turn in the third layer, in yellow. As before, the first turn in the third layer is located just “downstream” from the center of the solenoid, where it can best help accelerate the mote through the leading half of the solenoid. The cross-section of the coil is roughly a triangle.

The third batch ends with turn # 25 and is shown in green in the following diagram.

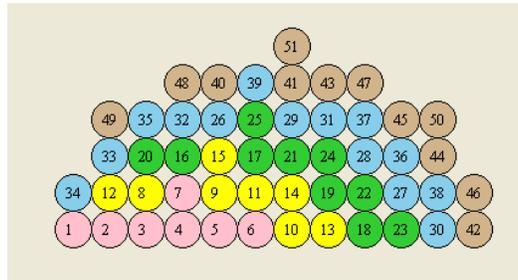


If you follow through the sequence in which the turns were added, it will be seen that the shape being built is, indeed, a triangle with its apex just downstream from the axial midpoint of the solenoid. The results for the next few batches are shown in the following diagrams.

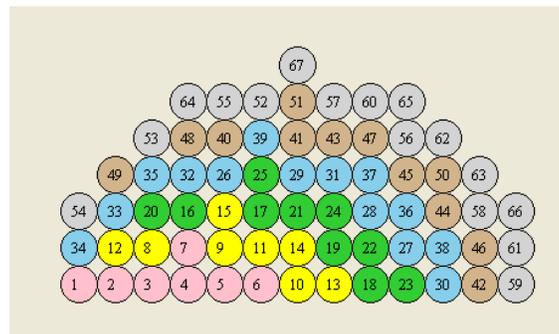
The fourth batch, ending at turn #39, is shown in here in blue. A new tendency seems to be developing, where the solenoid is being fleshed out beneath the triangle.



The fifth batch, ending at turn #51, is shown here in brown.



The sixth batch, ending at turn #67, is shown here in gray.



The conclusions to be drawn from this analysis – maximizing the work done on the mote – are:

1. The best coil is not a single-layer solenoid, but has a depth (outside radius less inside radius) which is comparable to the axial length of the coil;
2. The outer layers of the coil will have a triangular shape and
3. The aspect ratio of the wire assembly will depend on the relative values of the core radius and the wire-to-wire separation.

### Maximizing the work done per unit length of the coil

In the previous sections, we examined where turns should be added to a coil in order to maximize the total kinetic energy imparted to a magnetic mote. Maximizing the kinetic energy is an important goal for a coil gun, but it is not the only goal. What we really want to do is maximize the kinetic energy of the mote per unit length of the coil. For practical reasons, the length of the barrel must be kept finite. We want to use the available length as effectively as possible.

Let us define the length of the coil as its maximum extent in the axial direction. For a coil consisting of a single turn, the axial length is the diameter of the wire. The coil resulting from the first batch we created in the previous section, which had six turns side-by-side and a seventh turn in the second layer, has an axial length of was six wire diameters.

As before, we will make turn #1 the leading turn, with as small a diameter as possible. When we add the second turn, it surrounds turn #1. It has the same axial displacement as turn #1, but a radius which is just big enough so that turn #2 encompasses turn #1. This makes sense. Whether turn #2 is placed outside of turn #1, or behind turn #1, it will add approximately the same amount of work to the mote. But, placing turn #2 behind turn #1 doubles the axial length of the coil, and causes this configuration to impart approximately one-half of the work per unit length as does surrounding turn #1 with turn #2.

We find that turn #3 surrounds turn #2, and that turn #4 surrounds turn #3. The turns are creating a flat coil, where each new turn is added around the circumference of its predecessor.

This process does not stop. No matter how many turns are in the flat coil, the best place for the next turn is around the circumference of its predecessor. This will seem surprising until I point out something obvious. As the radii of the turns being added to the coil gets bigger and bigger, the contribution of each turn to the mote's total kinetic energy gets smaller and smaller. But, no matter how small the extra contribution is, it still increases the work per unit length. Adding a turn behind turn #1, on the other hand, will double the length of the coil and so cut the work per unit length in half.

The diameter of the finished coil will be subject to some constraints, just like its axial length. For example, the outermost turns add a disproportionate amount of Ohmic resistance compared with turn #1, a fact that will eventually force us to stop adding layers.

For the time being, let us simply limit the number of layers (for an unspecified reason) to, say, ten. The following diagram shows where the first 60 turns are placed when we maximize the work done per unit length, subject to a maximum of ten layers.



I have not shown the batches in color. Inspection shows that the best positioning is obtained when the leading turn is surrounded by nine additional turns. After that, the next ten best locations describe a second flat coil behind the first. The next best ten locations after that describe a third flat coil behind the second. And so on.

The conclusions to be drawn from this analysis – maximizing the work done on the mote per unit length of the coil – are:

1. Firstly, the best coil should have as many layers as other considerations permit and
2. Secondly, the axial length of the coil can then be established by other considerations.

Jim Hawley  
October 2012

An e-mail setting out errors and omissions would be appreciated.

## Appendix "A"

### Listing of the Visual Basic program

The following is a listing of the Visual Basic program named *AddTurnN*. The program was developed in Visual Basic 2010 Express, available as a free download from Microsoft Corporation. The program consists of a single form.

```
Option Strict On
Option Explicit On
```

```
' Add N turns - Analysis #1
```

```
Public Class Form
```

```
    Inherits System.Windows.Forms.Form
```

```
    ' Number of turns to consider
```

```
    Public N As Int32 = 67 ' Number of turns in the finished coil
```

```
    Public M As Int32      ' Index of the last turn fixed in place
```

```
    ' Dimensional variables using #20 guage wire, whose diameter = 0.812 mm
```

```
    Public Rwire As Double = 0.000406 ' Radius of wire
```

```
    Public Rcore As Double = 0.01     ' Core radius of coil
```

```
    Public Rmin As Double = Rcore + Rwire ' Minimum radius of all turns
```

```
    Public Rsep As Double = 2 * Rwire  ' Minimum separation of wire centerlines
```

```
    Public Zlead As Double = 0.2       ' Left abscissa of the leading turn
```

```
    Public zM As Double                ' Displacement of mote at any time
```

```
    ' Array of possible co-ordinates of the top centerlines
```

```
    Public RArray(N) As Double         ' Radius at N x N grid points
```

```
    Public ZArray(N) As Double         ' Axial displacement at N x N grid points
```

```
    Public FArray(N, N) As Boolean     ' True means occupied; False means unoccupied
```

```
    ' Index of co-ordinates occupied by the first M turns. For example, turn #1 has its
    ' centerline at RArray(RIndex(1)) and ZArray(ZIndex(1)), turn #2 has its centerline
    ' at RArray(RIndex(2)) and ZArray(ZIndex(2)), and so forth, up to the turn #M, which
    ' has its centerline at RArray(RIndex(M)) and ZArray(ZIndex(M)).
```

```
    Public RIndex(N) As Int32
```

```
    Public ZIndex(N) As Int32
```

```
    ' These two variables are used to reduce the size of the grid tested when adding the
    ' M + 1st turn. When M turns have been fixed in the coil, RMaxIndex and ZMaxIndex
    ' maximum values of the indices Ir and Iz in the grid which have been occupied. To
    ' find out where the M + 1st turn should be placed, it is only necessary to examine
    ' the rectangular grid defined by (Ir + 1) x (Iz + 1), not the full N x N grid. That
    ' is, the M + 1st turn will always be added so that it touches an existing turn.
```

```
    Public RMaxIndex As Int32
```

```
    Public ZMaxIndex As Int32
```

```
    ' Results
```

```
    Public W(N) As Double              ' Best value of work done when there are M turns
```

```
    Public Zstop(N) As Double          ' Ending value of zM when there are M turns
```

```
    ' Integration variables
```

```
    Public delZ As Double = 0.00001 ' Integration step size
```

```
    Public Zmax As Double = 2        ' Limits of integration
```

```
    Public Zmin As Double
```

```

Public Bz As Double          ' Some contribution to Bz
Public BzTotal1 As Double   ' The total of Bz contributions at z = zM
Public BzTotal2 As Double   ' The total of Bz contributions at z = zM + dz
Public dz As Double = 0.0000001
Public dBzdz As Double     ' Approximation of the derivative at z = zM
Public fzM As Double       ' Force per unit volume at z = zM

' Temporary storage as the M + 1st turn is examined at various positions
Public IrTest As Int32     ' r co-ordinate being tested for turn M + 1
Public IzTest As Int32     ' z co-ordinate being tested for turn M + 1
Public WTest As Double     ' Work for the M + 1st turn being tested
Public ZstopTest As Double ' Ending zM for the M + 1st turn being tested
Public IrBest As Int32     ' Best r co-ordinate for M + 1st turn found so far
Public IzBest As Int32     ' Best z co-ordinate for M + 1st turn found so far
Public WBest As Double     ' Maximum work for M + 1st turn found so far
Public ZstopBest As Double ' Ending zM for best M + 1st turn found so far

Public Sub New()
    InitializeComponent()
    With Me
        Name = ""
        Text = "Add N turns - Analysis #1"
        FormBorderStyle = Windows.Forms.FormBorderStyle.FixedSingle
        Size = New Drawing.Size(1024, 760)
        CenterToScreen()
        Visible = True
        Controls.Add(buttonExit)
        Controls.Add(buttonRun)
        Controls.Add(DisplayText)
        Controls.Add(DisplayDrawing)
        PerformLayout()
        BringToFront()
    End With
End Sub

Private WithEvents buttonExit As New Windows.Forms.Button With _
    {.Size = New Drawing.Size(80, 30), _
     .Location = New Drawing.Point(5, 5), _
     .Text = "Exit", .TextAlign = ContentAlignment.MiddleCenter}

Public Sub buttonExit_Click() Handles buttonExit.MouseClick
    Application.Exit()
End Sub

Private WithEvents buttonRun As New Windows.Forms.Button With _
    {.Size = New Drawing.Size(80, 30), _
     .Location = New Drawing.Point(90, 5), _
     .Text = "Run", .TextAlign = ContentAlignment.MiddleCenter}

Private Sub buttonRun_Click() Handles buttonRun.MouseClick
    MainProgram()
End Sub

Public DisplayText As New Windows.Forms.Label With _
    {.Size = New Drawing.Size(300, 700), _
     .Location = New Drawing.Point(5, 40), _
     .Text = "", .TextAlign = ContentAlignment.TopLeft}

```

```

Public DisplayDrawing As New Windows.Forms.Panel With _
    {.Size = New Drawing.Size(500, 500), _
     .Location = New Drawing.Point(310, 40), _
     .BorderStyle = BorderStyle.None}

Public DisplayBitmap As New Bitmap(1000, 500)

Private Sub MainProgram()
    '
    ' Specify the array of possible centerlines
    For I As Int32 = 1 To N Step 1
        RArray(I) = Rmin + ((I - 1) * Rsep)
        ZArray(I) = Zlead - ((I - 1) * Rsep)
    Next I
    '
    ' Clear the array of occupied grid locations
    For I As Int32 = 1 To N Step 1
        For J As Int32 = 1 To N Step 1
            FArray(I, J) = False
        Next J
    Next I
    '
    ' Specify the location of the first turn.
    RIndex(1) = 1
    ZIndex(1) = 1
    FArray(1, 1) = True
    RMaxIndex = 1
    ZMaxIndex = 1
    '
    ' Find the work and stopping zM for the first turn.
    IntegrateOneRun(_
        0, RIndex, ZIndex, _
        RArray(RIndex(1)), ZArray(ZIndex(1)), _
        W(1), Zstop(1))
    M = 1
    '
    ' Main loop to step through the turns 2 through N
    For Iturn As Int32 = 2 To N Step 1
        '
        ' Set the initial work for the M + 1st turn
        WBest = -1
        '
        ' Sub-loop to step through the available positions in the N x N grid
        For Ir As Int32 = 1 To (RMaxIndex + 1) Step 1
            For Iz As Int32 = 1 To (ZMaxIndex + 1) Step 1
                ' Only test this position if it is unoccupied
                If (FArray(Ir, Iz) = False) Then
                    '
                    ' Display progress on the monitor
                    DisplayText.Text = _
                        "Now working on turn #" & Trim(Str(Iturn)) & vbCrLf & _
                        "Testing co-ordinates Ir =" & Str(Ir) & _
                        " and Iz =" & Str(Iz)
                    DisplayText.Refresh()
                    '
                    ' Integrate the run with the M + 1st turn at the test location
                    IrTest = Ir
                    IzTest = Iz
                End If
            Next Iz
        Next Ir
    Next Iturn
End Sub

```

```

        IntegrateOneRun( _
            M, RIndex, ZIndex, _
            RArray(IrTest), ZArray(IzTest), _
            WTest, ZstopTest)
    ,
    ' If the work done is the greatest so far, then save it
    If (WTest >= WBest) Then
        IrBest = IrTest
        IzBest = IzTest
        WBest = WTest
        ZstopBest = ZstopTest
    End If
End If
'
' Give other processes a chance to work
Application.DoEvents()
Next Iz
Next Ir
'
' Ensure that some position was found
If (WBest < 0) Then
    MsgBox("No position was found for turn #" & Trim(Str(M + 1)), _
        MsgBoxStyle.OkOnly, "Add N turns")
    Exit Sub
End If
'
' Store the results for the M + 1st turn
M = M + 1
RIndex(M) = IrBest
ZIndex(M) = IzBest
W(M) = WBest
Zstop(M) = ZstopBest
FArray(IrBest, IzBest) = True
'
' Update the bounds of the grid to be examined for the next turn
If (IrBest > RMaxIndex) Then
    RMaxIndex = Math.Min(IrBest, N - 1)
End If
If (IzBest > ZMaxIndex) Then
    ZMaxIndex = Math.Min(IzBest, N - 1)
End If
'
' Display progress
Dim s As String = ""
For I As Int32 = 1 To M Step 1
    s = s & _
        "Turn #" & Str(I) & _
        " has Ir = " & Trim(Str(RIndex(I))) & _
        " Iz = " & Trim(Str(ZIndex(I))) & _
        " r = " & Str(RArray(RIndex(I))) & _
        " z = " & Str(ZArray(ZIndex(I))) & vbCrLf
Next I
DisplayText.Text = s
DisplayText.Refresh()
Threading.Thread.Sleep(2000)
'
' Continue and place the next turn
Next Iturn

```

```

'
' Display results
Dim ss As String = "All finished with: " & vbCrLf & _
    "Rcore =" & Str(Rcore) & " and Rwire =" & Str(Rwire) & vbCrLf
For I As Int32 = 1 To N Step 1
    ss = ss & _
        "Turn #" & Str(I) & _
        " has Ir = " & Trim(Str(RIndex(I))) & _
        " Iz = " & Trim(Str(ZIndex(I))) & _
        " r =" & Str(RArray(RIndex(I))) & _
        " z =" & Str(ZArray(ZIndex(I))) & vbCrLf
Next I
DisplayText.Text = ss
DisplayText.Refresh()
'
' Render the results in DisplayBitmap. The cross-section of the upper half of
' the completed coil is shown. Each turn is represented by circle 30 pixels in
' diameter. Inside each such circle is the index in the sequence by which that
' turn is added.
Dim g As Graphics = Graphics.FromImage(DisplayBitmap)
Dim ThisPen As New Drawing.Pen(Color.Black, 1)
Dim ThisBrush As New Drawing.SolidBrush(Color.Black)
Dim ThisFont As New Font("Times New Roman", 10)
Dim CircleRad As Double = 15 ' Radius of rendered circle, in pixels
Dim SF As Double = Rwire / CircleRad ' Scale factor, in meters per pixel
Dim CircleMidX As Double ' x co-ordinate of center of circle, in meters
Dim CircleMidY As Double ' y co-ordinate of center of circle, in meters
Dim CircleLeft As Double ' x co-ordinate of left of circle, in pixels
Dim CircleTop As Double ' y co-ordinate of top of circle, in pixels
For Iturn As Int32 = 1 To N Step 1
    CircleMidX = -(ZArray(ZIndex(Iturn)) - Zlead) + Rwire
    CircleMidY = RArray(RIndex(Iturn)) - Rmin + Rwire
    CircleLeft = (CircleMidX - Rwire) / SF
    CircleTop = (CircleMidY + Rwire) / SF
    ' Select color based on results of previous runs
    If ((Iturn >= 1) And (Iturn <= 7)) Then
        ThisBrush = New Drawing.SolidBrush(Color.Pink)
    End If
    If ((Iturn >= 8) And (Iturn <= 15)) Then
        ThisBrush = New Drawing.SolidBrush(Color.Yellow)
    End If
    If ((Iturn >= 16) And (Iturn <= 25)) Then
        ThisBrush = New Drawing.SolidBrush(Color.LimeGreen)
    End If
    If ((Iturn >= 26) And (Iturn <= 39)) Then
        ThisBrush = New Drawing.SolidBrush(Color.SkyBlue)
    End If
    If ((Iturn >= 40) And (Iturn <= 51)) Then
        ThisBrush = New Drawing.SolidBrush(Color.Tan)
    End If
    If ((Iturn >= 52) And (Iturn <= 67)) Then
        ThisBrush = New Drawing.SolidBrush(Color.LightGray)
    End If
    g.FillEllipse(ThisBrush, _
        CInt((0.05 * DisplayBitmap.Width) + CircleLeft), _
        CInt((0.95 * DisplayBitmap.Height) - CircleTop), _
        CInt(2 * CircleRad), _
        CInt(2 * CircleRad))

```

```

g.DrawEllipse(ThisPen, _
  CInt((0.05 * DisplayBitmap.Width) + CircleLeft), _
  CInt((0.95 * DisplayBitmap.Height) - CircleTop), _
  CInt(2 * CircleRad), _
  CInt(2 * CircleRad))
ThisBrush = New Drawing.SolidBrush(Color.Black)
g.DrawString(Trim(Str(Itern)), _
  ThisFont, ThisBrush,
  CSng((0.05 * DisplayBitmap.Width) + CircleLeft + 7), _
  CSng((0.95 * DisplayBitmap.Height) - CircleTop + 8))
Next Itern
g.Dispose()
' Render the circles
DisplayDrawing.BackgroundImage = DisplayBitmap
DisplayDrawing.Refresh()
End Sub

```

```

'/////////////////////////////////////////////////////////////////
' Sub IntegrateOneRun is a general subroutine which integrates one run, where the
' mote starts at z = zM. The current is stopped whenever the force turns repulsive.
' The coil consists of M + 1 turns, where the first M turns are fixed in position and
' the run is being done to help decide where the M + 1st turn should be placed. This
' subroutine can handle the first turn, for which M = 0.
' The arguments are:
'   M = number of turns already fixed in place
'   Rindex(i = 1 to M) holds the indices of the radii of the first M turns
'   Zindex(i = 1 to M) holds the indices of the displacements of the first M turns
'   rNext is the radius of the M + 1st turn
'   zNext is the axial displacement of the M + 1st turn
' The return values, as ByRef arguments, are:
'   W is the work done
'   Zstop is the axial displacement when the current is stopped
Private Sub IntegrateOneRun( _
  ByVal M As Int32, ByVal Rindex() As Int32, ByVal Zindex() As Int32, _
  ByVal rNext As Double, ByVal zNext As Double, _
  ByRef W As Double, ByRef Zstop As Double)
  Dim zMzturn As Double
  Dim zMzturn_rturn As Double
  ' Set zM to its starting value
  zM = Zmax + delZ
  ' Initialize the work
  W = 0
  Do
    ' Increment to the next axial displacement
    zM = zM - delZ
    ' Calculate Bz at zM and store in BzTotal1
    BzTotal1 = 0
    If (M > 0) Then
      For Im As Int32 = 1 To M Step 1
        zMzturn = zM - ZArray(Zindex(Im))
        zMzturn_rturn = (zMzturn * zMzturn) + _
          (RArray(Rindex(Im)) * RArray(Rindex(Im)))
        Bz = zMzturn_rturn * _
          RArray(Rindex(Im)) * RArray(Rindex(Im)) * _
          RArray(Rindex(Im)) * RArray(Rindex(Im)) / _
          (zMzturn_rturn ^ 4)
        BzTotal1 = BzTotal1 + Bz
      Next Im
    End If
    ' Increment to the next axial displacement
    zM = zM + delZ
    ' Calculate Bz at zM and store in BzTotal1
    BzTotal1 = 0
    If (M > 0) Then
      For Im As Int32 = 1 To M Step 1
        zMzturn = zM - ZArray(Zindex(Im))
        zMzturn_rturn = (zMzturn * zMzturn) + _
          (RArray(Rindex(Im)) * RArray(Rindex(Im)))
        Bz = zMzturn_rturn * _
          RArray(Rindex(Im)) * RArray(Rindex(Im)) * _
          RArray(Rindex(Im)) * RArray(Rindex(Im)) / _
          (zMzturn_rturn ^ 4)
        BzTotal1 = BzTotal1 + Bz
      Next Im
    End If
    ' Stop if the force is repulsive
    If (BzTotal1 < 0) Then
      Zstop = zM
      Exit Do
    End If
    ' Calculate the work done
    W = W + (CArray(Cindex) * zM) * (BzTotal1 * delZ)
  Loop
End Sub

```

```

End If
zMzturn = zM - zNext
zMzturn_rturn = (zMzturn * zMzturn) + (rNext * rNext)
Bz = zMzturn_rturn * rNext * rNext * rNext * rNext / _
    (zMzturn_rturn ^ 4)
BzTotal1 = BzTotal1 + Bz
' Calculate Bz at zM + dz and store in BzTotal2
BzTotal2 = 0
If (M > 0) Then
    For Im As Int32 = 1 To M Step 1
        zMzturn = zM + dz - ZArray(Zindex(Im))
        zMzturn_rturn = (zMzturn * zMzturn) + _
            (RArray(Rindex(Im)) * RArray(Rindex(Im)))
        Bz = zMzturn_rturn *
            RArray(Rindex(Im)) * RArray(Rindex(Im)) * _
            RArray(Rindex(Im)) * RArray(Rindex(Im)) / _
            (zMzturn_rturn ^ 4)
        BzTotal2 = BzTotal2 + Bz
    Next Im
End If
zMzturn = zM + dz - zNext
zMzturn_rturn = (zMzturn * zMzturn) + (rNext * rNext)
Bz = zMzturn_rturn * rNext * rNext * rNext * rNext / _
    (zMzturn_rturn ^ 4)
BzTotal2 = BzTotal2 + Bz
' Calculate the derivative
dBzdz = (BzTotal2 - BzTotal1) / dz
' Calculate the force per unit volume
fzM = BzTotal1 * dBzdz
' Add the incremental work done
If (fzM < 0) Then
    W = W - (fzM * delZ)
Else
    Zmin = zM + delZ
    Exit Do ' The force per unit volume has turned positive.
End If
Loop
End Sub

End Class

```