Physics of a tractor pull, run by a 2WD pickup truck

We all love a good truck pull contest. It is a lovely display of some of the basic principles of physics. There is also the noise, action, smoke and, of course, the beer. In this paper, I want to examine some of these principles to see what they can tell us about winning.

Part I -- Simple mathematical models for the truck and sled

To get right to the heart of the matter, we need first to construct mathematical models for the truck and the sled. The best place to start is with the simplest model which captures the important characteristics of the system. Even a crude model can shed light on which factors are most important; less significant features can always be added later.

The following figure shows where I started, with the rig viewed from the side. The direction of travel is towards the right. The truck is represented by its two sets of wheels and by a thin platform. I have rendered the truck in blue and the sled, which is dragged behind, in red. At the front end of the sled is the "pan", which rests on the ground. The sled's rear end is supported by an axle with wheels. The black bar is the tow chain or tow. If it is a chain, we will assume that the tension in the chain keeps it taut and, therefore, straight. In this simple model, the hitch points are intended to be thought of as ball hitches, which we will assume are frictionless.

One very important simplification made in this model is the placement of all the significant parts at a constant height above ground. The deck of the sled passes through the center of its rear axle. The chassis of the truck passes through the centers of its front and rear axles. The sled's deck and the truck's chassis are the same height above ground. The hitch points are at this height, too, so the tow chain is horizontal.

Placing everything at the same height above ground eliminates many of the mechanical moments which occur when the tow chain is not horizontal. Moments (the mechanical kind, that is) tend to rotate objects in the same way that forces tend to push them in straight lines. When they exist, these moments cause changes in the effective distribution of the weight among the wheels and the pan. In due course, we will relax this assumption, and examine the situation when the hitch points are at different heights.

Let's define five masses to account for the weight of the vehicles. These masses are shown in the following figure, superimposed on the basic framework at the points where they are presumed to act.
The first subscript identifies the vehicle, $s$ for the sled and $t$ for the truck. The second subscript identifies the placement along the vehicle, $f$ for front, $r$ for rear and, in the case of the sled, $m$ for the moveable weight. For example, $M_{tf}$ is the effective mass over the truck's rear axle.

The locations of four of the masses, those other than the moveable weight, are fixed. For convenience, we will assume that the masses can be treated as concentrated points, whose locations are the following:

- $M_{tf}$ is the mass concentrated at the center of the front axle of the truck,
- $M_{tr}$ is the mass concentrated at the center of the rear axle of the truck,
- $M_{sf}$ is the mass concentrated at the vertical center-line of the pan,
- $M_{sm}$ is the mass of the moveable weight and
- $M_{sr}$ is the mass concentrated at the center of the rear axle of the sled.

The sum of the first two masses, $M_{tf} + M_{tr}$, is the total mass of the truck. The total mass can be apportioned between the axles depending on the front-to-rear loading ratio. We will treat the truck's masses as being constant although, in practice, the mass will decrease slightly as fuel and oil are burned during a run.

The sum of the latter three masses, $M_{sf} + M_{sm} + M_{sr}$, is the total mass of the sled. We will treat these three masses as being constant, as well. The apportionment between the front and rear masses of the sled, $M_{sf}$ and $M_{sr}$, respectively, is determined by the front-to-rear loading ratio which obtains when the moveable weight is removed from the sled.

We need some dimensions, which are shown on the following figure.

- $W_t$ is the wheelbase of the truck,
- $W_s$ is the "wheelbase" of the sled, being the distance between the vertical center-line of the pan and the rear axle,
- $W_h$ is the "wheelbase of the hitch", being the distance from the truck's rear axle to the vertical center-line of the pan. This is not intended to be merely the length of the chain,
- $D_w$ is the distance at any particular time, measured towards the front of the sled, from the rear axle of the sled to the moveable weight and
- $H_s$ is the height of the sled's deck which, in the simplified model, is the only height considered.

All of these dimensions remain constant during a run except for $D_w$. During a run, the operator of the sled causes the moveable weight to move forward at a constant rate. That increases the effective weight on the pan, and so increases the friction which the pan experiences as it is dragged along the ground. This is a good time to introduce the parameters we need to describe the location of the moveable weight as it
changes with time. Since $D_w$ changes with time, it is appropriate to express its dependence on time explicitly. Let's use the symbol $t$ for time, measured in seconds from time $t = 0$ at the start of a run. Let's use the symbol $R_w$ as the rate at which the moveable weight moves forward. $R_w$ will have the units of meters per second. Lastly, let's use the symbol $D_{w0}$ for the distance from the rear axle to the moveable weight at the start of a run. Then, we can express the distance $D_w$ as the following function of time:

$$D_w = D_{w0} + R_w t \quad (1)$$

We are now going to consider the truck and sled separately. The only connection between them is the tow chain which, given our assumptions, is always parallel to the ground. Let's use the symbol $T$ for the instantaneous tension in the tow chain. "Tension" is one type of force. There are lots of different kinds of forces, such as the the force of gravity, the drag force exerted on the pan, electrostatic forces, and so on. The tension in the tow chain is not constant but will vary during the course of a run. In accordance with one of Newton's Laws, every force is accompanied by an equal and opposite force. In our case, the force which drags the sled forwards, towards the right, is counteracted by a force with exactly the same magnitude but which tends to retard the forward progress of the truck. The tension tends to pull the chain apart. The relationship between these two equal and opposite forces is shown in the following figure.

We can, in fact, remove the tow chain as a physical object. The physics of the vehicles' motion will be preserved if we replace the tow chain with the equal and opposite forces I have just described, thus:

Next, let's deal with the force of gravity. We will use the symbol $g$ for the local gravitational acceleration. In S.I. units, $g$ is approximately equal to 9.81 meters per second per second, or 9.81 m/s$^2$. The gravitational force acting on a object which has mass $m$ is the product $mg$. If the mass is given in kilograms, then the force will have units of Newtons. Because we have lined up the rig horizontally with respect to the surface of the ground, we know that the forces exerted by the Earth on our five masses will act straight downwards. In the following figure, the five masses have been replaced by their equivalent gravitational forces, which are rendered in orange. To avoid clutter, I have omitted the horizontal dimension marks.
Now, let's deal with the ground. Under the influence of the force of gravity, a wheel, for example, pushes down on the ground. The ground pushes back upwards with a force having equal magnitude. In our simplified model, we will assume that each set of wheels is perfectly round and makes contact with the ground at a single point directly below the center of its axle. Real wheels, of course, have less than infinite internal pressure and so make contact with the ground over a small patch. It is enough for our preliminary analysis to idealize the contact to a single point. We will make the same assumption for the pan. Even though the pan contacts the ground over a relatively large surface area, we will assume that the responding force exerted upwards by the ground acts at a single point.

The pan and the rear wheels of the truck differ from the other two sets of wheels. Let me describe the pan first. It provides drag. In fact, that is the whole purpose of the sled. The drag expresses itself as friction when the pan is dragged along the surface. A typical mathematical model for the frictional drag on an object like the pan is shown in the following figure.

\[ F_d = C_{drag} F_L \]

The object is being dragged along a surface. It is moving, not stopped. The object is pressed against the surface by some force \( F_L \). This force is not necessarily gravity, but could arise from a variety of other sources, such as magnetic attraction. The subscript \( \perp \) is the mathematical symbol for "perpendicular" and it should be understood that force \( F_L \) is the force acting perpendicularly to the surface.

The drag force \( (F_d) \) is a retarding force, which means it acts in the direction opposite to the direction of motion of the object. The magnitude of the drag force is the product of the perpendicular force multiplied by a constant coefficient of drag \( C_{drag} \). This constant depends on such factors as the roughness of the surfaces of the object and ground where they make contact. It does not depend on the area of contact and, even more importantly, it does not depend on the speed with which the object is moving. The coefficient \( C_{drag} \) will always have a value somewhere between zero and one. If the surface is completely frictionless, the coefficient is zero and the drag force will be zero. If the surface is extremely rough, the constant may get close to one, which means that the force needed to keep the object moving is almost as much as the perpendicular force.

It is important to understand that the drag coefficient is constant so long as the object keeps moving. If the object comes to a stop, or is at rest to begin with, the friction which must be overcome to get it moving is somewhat greater than the friction it experiences once it gets underway. Usually, a second drag coefficient is given, which applies only when the object is at rest. We are not going to use this second drag coefficient in our simplified model. We will assume that the truck has sufficient power to get the object (that is, the sled) moving within a time period which is negligible compared with the duration of a run.

To apply this concept to the sled, we need to know the force with which the pan presses down on the ground. It is important to understand that the total force with which the pan presses down on the ground is not just the force of gravity acting on mass \( M_{ef} \) which is the concentrated mass assumed to lie right over the pan. It also includes a contribution from the force of gravity acting on the moveable weight. Not just that, but the force is also affected by a "dynamic" load caused by torques acting on the sled. Of course, whatever the force turns out to be, the ground will back upwards on the pan with exactly the same
force. Let’s use the symbol $F_{sf}$ for this force where, as before, the first subscript identifies the sled and the second identifies the front of the sled. In a similar way, we can define symbols for the forces which the ground exerts on the rig at all four points of contact. In addition, we are going to need a symbol for the drag on the pan. The new variables are:

- $F_{sf}$ is the vertical force which the ground exerts on the front wheels of the truck,
- $F_{tr}$ is the vertical force which the ground exerts on the rear wheels of the truck,
- $F_{sr}$ is the vertical force which the ground exerts on the center-point of the sled's pan,
- $F_{rr}$ is the vertical force which the ground exerts on the rear wheels of the sled and
- $D_{sf}$ is the horizontal drag force which the ground exerts on the bottom of the pan.

As we have done before, we can now remove the ground from the picture, and replace its effect with the forces which it exerts on the two vehicles. See this:

![Diagram](image)

Just to be clear, the vertical force which acts between the ground and the pan is equal to $F_{sf}$. This force acts perpendicularly to their mutual contact surface. Since we will encounter other coefficients of friction and drag, I will not use $c_{drag}$ for the coefficient of drag of the pan. Instead, I will use $c_{pd}$, where the subscripts stand for "pan, dynamic". Then, we can express the drag force on the pan as follows:

$$D_{sf} = c_{pd}F_{sf} \quad (2)$$

Note that all five of these new forces act at points along the ground, rather than acting at points at deck height. (Generally speaking, when an arrow is drawn to represent a force, the tail of the arrow is presumed to be located at the point at which the force acts of the object.)

Note also that I have left a little bit of the ground showing beneath the drive wheels of the truck. I mentioned above that these drive wheels are not exactly the same as the front wheels of the truck and the rear wheels of the sled. The front wheels of the truck and the rear wheels of the sled are not powered. They merely turn at whatever speed is needed to permit the vehicle to move. In reality, there is some friction, or retarding torque, as these wheels turn, but we will ignore it in the simplified model. In essence, the front wheels of the truck and the rear wheels of the sled are nothing more than a mechanism which allows the two vehicles to move without experiencing undue friction.

**Part II -- The reactions at the drive wheels**

The drive wheels of the truck are different. The truck's engine forces them to turn. Whether or not the truck moves forward when the drive wheels turn depends on the interaction between the wheels and the ground. In this section, we will look into this matter, beginning with the following figure. For the sake of clarity, I have left a small gap between the bottom of the wheel and the ground surface. The wheel has radius $R$. (Careful readers will realize that, in the simplified model, the radius $R$ of these wheels is, by definition, exactly the same as the deck height $H_S$ which applies to both vehicles.) The axle, which is shown as the heavy black dot in the figure, is secured to the truck's chassis.
When the truck is pulling the sled, its engine turns the crankshaft. Since the crankshaft will be turning at a higher angular speed (measured in RPM or in radians per second) than is suitable for the wheels, a transmission is inserted between the crankshaft and the driveshaft. The transmission reduces the angular speed to a more appropriate rate. We can imagine the transmission to be a set of gears, or perhaps even a single pair of gears. Power is transmitted from the shaft driving the "input" gear to the shaft connected to the "output" gear. If the transmission is perfect, the torque which the crankshaft delivers to the input gear is transmitted without loss to the driveshaft connected to the output gear. In reality, of course, the transmission will not be perfect and there will be a reduction of power as it passes through the transmission.

Power is preserved when passing through a gear mesh, but rotational speed and torque are not. Rotational speed and torque are affected in reciprocal ways. If the gears have teeth, the mesh is characterized by the ratio of the number of teeth on the output gear divided by the number of teeth on the input gear. This quotient is called the "gear ratio". If the gears are implemented using two sheaves (or pulleys) and a belt, the ratio of the diameter of the output pulley divided by the diameter of the input sheave is the gear ratio. A gear ratio greater than one means the output gear is bigger around than the input gear. To get the bigger, output, gear to rotate one full revolution will require the input gear to make more than one revolution. It follows that the output shaft will rotate more slowly than the input shaft.

Let's consider what happens to torque. Torque is transferred from the input gear to the output gear through a force which acts in a direction tangent to the circumferences of both gears at the point where they touch. The torque applied to the input gear generates the tangential force. The tangential force is transmitted to the output gear, where it produces a corresponding torque on the output shaft. The torques on both shafts are related to the interaction force by the standard formula: torque equals force multiplied by lever-arm. The lever-arm of each gear is its radius which is, of course, proportional to its diameter. If the output gear has the larger radius, it will generate more torque from the interaction force than the torque on the input gear which generated the interaction force.

If we let $GR$ be the gear ratio, then the relationship between the rotational speeds and the torques on either side of a gear mesh is:

$$\text{Rotational speed}_{\text{out}} = \frac{\text{Rotational speed}_{\text{in}}}{GR}$$  \hspace{1cm} (3A)

$$\text{Torque}_{\text{out}} = GR \times \text{Torque}_{\text{in}}$$  \hspace{1cm} (3B)

I mentioned the loss of power flowing through a gear mesh. Such losses can be quantified by stating the efficiency of the gear mesh. We will use the symbol $\eta$, which is the greek letter "eta", for efficiency. It
will always be a fraction somewhere between zero and one. For a transmission, it will probably be about 0.95. When the clutch is fully engaged, the loss of power is generally the result of loss of torque, not of loss of rotational speed. In our simplified model, we will describe the transmission as follows:

\[
\text{Rotational speed}_{\text{out}} = \frac{\text{Rotational speed}_{\text{in}}}{GR_{\text{trans}}} \quad (4A)
\]

\[
\text{Torque}_{\text{out}} = \eta_{\text{trans}} \times GR_{\text{trans}} \times \text{Torque}_{\text{in}} \quad (4B)
\]

The driveshaft rotates around an axis which runs more or less down the longitudinal center-line of the truck. Where the driveshaft meets the rear axle, a differential is used to change the axis of rotation so that it now runs across the width of the truck. A typical differential will contain three gear meshes. The differential does more than just change the primary direction of rotation. It also includes a facility which permits the wheels on either end of the axle to rotate at different speeds, as they are wont to do when the vehicle turns a corner. Virtually all differentials also provide another step down in the rotational speed, so the axle rotates at an even lower speed than the driveshaft. If we let \( \eta_{\text{diff}} \) and \( GR_{\text{diff}} \) be the efficiency and gear ratio of the differential, respectively, then we can describe the performance of the differential in precisely the same way as did for the transmission:

\[
\text{Rotational speed}_{\text{out}} = \frac{\text{Rotational speed}_{\text{in}}}{GR_{\text{diff}}} \quad (5A)
\]

\[
\text{Torque}_{\text{out}} = \eta_{\text{diff}} \times GR_{\text{diff}} \times \text{Torque}_{\text{in}} \quad (5B)
\]

It is common to use the symbol \( \tau \), which is the Greek letter "tau", for torques. The torque which is applied to the rear wheels of the truck is shown in the figure as the red circular arrow labeled \( \tau_{\text{twr}} \). If we let \( \tau_{\text{crank}} \) be the torque at the crankshaft, then:

\[
\tau_{\text{twr}} = \eta_{\text{diff}} GR_{\text{diff}} \eta_{\text{trans}} GR_{\text{trans}} \tau_{\text{crank}} \quad (6)
\]

Just like forces, torques are always associated with equal and opposite torques. In the case at hand, the body of the truck (which includes the engine, transmission, driveshaft and differential) exerts a clockwise torque of magnitude \( \tau_{\text{twr}} \) on the rear wheels. They in turn exert a counter-clockwise torque of magnitude \( \tau_{\text{twr}} \) on the body of the truck. In some circumstances, this counter-clockwise torque results in the body of the truck rotating counter-clockwise around its rear axe. The truck is said to "perform a wheelie".

The torque \( \tau_{\text{twr}} \) tends to rotate the tire. At the point where it makes contact with the ground, the tread on the tire tends to move towards the left, towards the rear of the truck. Since the weight over the rear wheels causes the tire to bear down on the ground surface, the tread will exert a horizontal force on the ground, which force also acts towards the rear of the truck. This force is represented in the figure above by the blue arrow labeled \( F_{w\rightarrow g} \), where the subscript \( w \rightarrow g \) denotes wheel-on-the-ground. If (and this is an important "if") the tread does not slip with respect to the ground, then the patch of ground underneath the point of contact will exert an equal and opposite force on the tread. This horizontal force, of the ground-on-the-wheel, is represented by the green arrow labeled \( F_{g\rightarrow w} \).

As I say, if the wheel does not slip, then these two forces are equal: \( F_{g\rightarrow w} = F_{w\rightarrow g} \). Sometimes, though, the wheel does slip. The ground cannot resist as much force as the tread wants to exert. If the ground can only resist some of the tread's force, what happens to the difference? It begins to accelerate the wheels. They begin to turn at a rate which is no longer consistent with the truck's forward speed. The wheels are slipping.
In the next few paragraphs, I want to describe three issues which will help us quantify the interaction between the tread and the ground. They are: (i) calculating the force of the wheel-on-the ground before slipping starts, (ii) determining when slipping will start and (iii) calculating the force of the wheel-on-the-ground after slipping starts. In our simplified model, we are going to assume that there is no slipping. Even so, this seems like a good time to introduce some of the factors we will need to take into account when we allow for slipping.

Issue #1 -- The force of the wheel-on-the-ground

When there is no slippage, all of the torque applied to the drive wheels is transmitted into the force of the wheel-on-the-ground. The torque and force are related by the standard formula: torque equals force multiplied by lever-arm. For the wheel, the lever-arm (commonly also referred to as the moment-arm) is the radius of the wheel. Using the symbols we have already defined, we can write this as:

\[ \tau_{twr} = RF_{w} - g \]  

(7)

If the radius is measured in feet and the force in pounds, then the units of torque will be foot-pounds. In S.I. units, the radius is measured in meters and the force in Newtons, so the units of torque are Newton-meters.

Issue #2 -- The onset of slippage

Equation (7) applies only if there is no slippage. We need some way to determine when slippage will start. For this purpose, we will examine the coefficient of friction between the wheel and the ground. The following figure shows a typical mathematical model for the frictional drag on an object when it is at rest. It is very similar to the figure shown above but, in this case, the object is not moving.

As before, \( F_{\perp} \) is the force which the object and the surface exert on each other across their common boundary. Some force is applied to the object in the direction parallel to the surface. In the figure, this is labelled the "Applied force". When the applied force is small, friction between the bottom of the object and the surface will prevent the object from moving. Like the drag on the pan, the force of friction is a retarding force, which acts in the direction opposite to the applied force. The magnitude of the force of friction will be determined automatically (by nature, to be more precise) so that it is just equal to the magnitude of the applied force. Since the two forces acting on the object are equal and opposite, the object will not accelerate along the surface. It will remain at rest. (Aside: If the applied force is applied at a point somewhere on the object other than at ground level, then the applied force and the frictional force are not colinear. They will exert a moment on the object. If the moment is great enough, it will topple the object over on its side. As a second aside: The words "torque" and "moment" are often used interchangeably. Indeed, from the point-of-view of physics, they are exactly the same concept. However, it is best practice to use the word "moment" when the object remains stationary and to use the word "torque" if the object moves.)
There is a maximum amount of retarding force which friction can generate. If the applied force exceeds this maximum, then the object will break away from rest, and begin accelerating in the direction of the applied force. In the figure, this maximum amount of retarding force is labeled $F_{\text{friction, max}}$.

$F_{\text{friction, max}}$ is calculated in the same way as the force of sliding friction we looked at above, namely, as a coefficient multiplied by the perpendicular force. The coefficient we are looking at here is called the coefficient of static friction to distinguish it from the coefficient of sliding friction (more commonly referred to as the coefficient of dynamic friction) which applied to the moving pan.

As we do our calculations, we will proceed as follows. We will use Equation (7) to calculate the force of the wheel-on-the-ground. But, we will keep an eye on the value of $C_{\text{friction}}F_{\perp}$. If the force calculated using Equation (7) is ever found to exceed $C_{\text{friction}}F_{\perp}$, then we know that the wheel has started to slip.

**Issue #3 -- The continuation of slippage**

Once the drive wheels start to slip, the torque-force relationship in Equation (7) is no longer valid. Not all of the torque applied to the wheels is absorbed by the ground. In addition, the maximum retarding force of static friction $(C_{\text{friction}}F_{\perp})$ no longer relevant, because the two surfaces are now slipping with respect to one another. We need something completely different to deal with things once slipping starts.

That something else is the coefficient of dynamic friction between the tread and the ground. Once slipping starts, the relationship between the tread and the ground is fundamentally the same as the relationship between the pan and the ground. Once again, we need a drag coefficient, which we will call $C_{\text{twr,d}}$, where the subscripts stand for “truck wheel rear, dynamic”. If $F_{\perp}$ is, as before, the vertical force between the wheel and the ground, then the horizontal force of friction of the wheel-on-the-ground will have the magnitude $C_{\text{twr,d}}F_{\perp}$.

Note that this force is completely unrelated to the torque applied by the engine to the wheels. It depends only on the distribution of effective weight among the rig’s supports and on the nature of interaction between treaded rubber and dirt. Furthermore, this force $(C_{\text{twr,d}}F_{\perp})$ will always be less than the force which sets the threshold of static friction $(C_{\text{friction}}F_{\perp})$. The coefficient of dynamic friction $C_{\text{friction}}$ will always be less than the coefficient of static friction $C_{\text{friction}}$ between corresponding surfaces. (It is easier for a moving steel or rubber plate, which already has forward momentum, to continue rotating little pieces of dirt on the ground, or jostling them out of each other’s way, than it is for a stopped plate to start the process from a standing start.) In any event, the horizontal forces exchanged between the tread and the ground will always be less after slipping starts than the maximum frictional force which determined the commencement of slipping.

As I have said, our simplified model will assume that the wheels do not slip. We can therefore use Equation (7) to calculate the force of the wheels-on-the-ground. We can assume that the ground will respond in kind. We get:

$$F_{g\rightarrow w} = \frac{\tau_{\text{twr}}}{R} \quad (8)$$

Since the rear wheels try to turn, but their bottoms are held in place by the ground, another set of forces arises. The wheels will try to roll forward, but the axle will be restrained by the fasteners which bolt it to the truck’s body. The axle will exert a horizontal force on the truck and, like always, the truck will exert an equal and opposite force on the axle. In a manner consistent with previous usage, we will use the
symbols $F_{a\rightarrow t}$ for the force of the axle-on-the-truck and $F_{t\rightarrow a}$ for the force of the truck-on-the-axle. It is handy at this point to consider the rear wheels separately from the rest of the truck, as is shown in the following figure.

Let's just focus on the rear wheels for the time being. The figure does not show the forces which act on the wheels in the vertical direction. When we looked at the rear wheels in Part I above, they were an integral part of the truck. We assigned a single mass, $M_{tr}$, to the rear axle. Now that we have separated the rear wheels from the rest of the truck, we really should divide this mass into a wheel part and a chassis part. We would then have a separate gravitational force on each sub-mass. The force of the ground pushing up on the bottom of the wheel would stay the same, but we would have to introduce two internal forces, for the chassis bearing down on the axle and for the axle pushing up on the chassis. That is a lot of vertical forces. Fortunately, we can ignore the vertical direction entirely for the time being. Experience tells us that the forces in the vertical direction will balance each other out, and that there will be no net upwards or downwards vertical force on either the wheels or the rest of the truck. If the forces did not balance each other out, so that the net vertical force was not equal to zero, the wheels or the chassis or both would accelerate upwards or downwards. Since they do not accelerate upwards or downwards, at least to any sustained degree, we can assert that the net vertical force and the vertical acceleration are both zero. (On very short time scales, the truck will hit bumps on the course, which cause bouncing. Bouncing is a manifestation of vertical acceleration. But these bumps are small enough that we will ignore them.)

The first thing we are going to do is add up the forces which act on the wheels in the horizontal direction. Forces which act towards the right are added to the sum. Forces which act towards the left are subtracted. If the total adds up to a positive number, we know the net force acts towards the right. If the total is a negative number, the net force acts towards the left. The sum of the horizontal forces on the rear wheels is:

$$\sum \text{HorizontalForces} = F_{g\rightarrow w} - F_{t\rightarrow a} \quad (9)$$

The fancy E, which is the Greek capital letter "sigma", is used to denote the sum of certain quantities.

The second thing we are going to do is add up the torques which cause the wheel to rotate. There are two torques. Engine power applies torque $\tau_{twr}$. The other torque is induced because the force of the ground-on-the-wheels does not act through the center of the axle. We are going to pick the center of the axle as the axis around which we want to measure the torques. The magnitude of the torque exerted by the force of the ground-on-the-wheels is the force multiplied by the lever-arm or moment-arm, which is equal to the radius $R$ of the wheel. Torques which act clockwise are added to the sum; torques which act counterclockwise are subtracted from the sum. A positive or negative total simply means that the net torque acts in the clockwise or counter-clockwise direction, respectively. The sum of the torques on the rear wheels, measured around the rear axle, is:
All right, now that we have expressions for the net horizontal force and the net torque, what next? This is where we start to use Newton's Law for dynamics, which is famously stated as:

\[
\text{Force} = \text{Mass} \times \text{Acceleration}
\]

A point-sized mass will accelerate at a rate equal to the applied force divided by the mass. The direction of the acceleration will be the same as the direction of the applied force. The same equation can be used without any modification for rigid bodies as well (which are too big to be approximated as point masses) if the force is applied at the center-of-mass, or center-of-gravity, of the rigid body. Newton's Law can also be expanded to deal with rotations. Conceptually, one can think of a rigid body as being comprised of a lot of very small pieces. By applying Newton's Law to the individual pieces, and then adding up the results, one gets, for circular motion:

\[
\text{Torque} = \text{Moment of inertia} \times \text{Angular acceleration}
\]

A torque is the rotational equivalent of a force. Similarly, a moment of inertia is the rotational equivalent of a mass. Thirdly, an angular acceleration is the rotational equivalent of a (linear) acceleration. When one speaks about an acceleration, without using any qualifier, it is assumed to be a linear, or straight-line, acceleration.

An acceleration is the rate at which the (linear) speed of an object increases. An angular acceleration is the rate at which the rotational speed of an object increases. Like acceleration, when one speaks about a speed, without using any qualifier, it is assumed to be a linear, or straight-line, speed.

Speed is often measured in MPH, or miles per hour. In this paper, we will use S.I. units for speed: meters per second. Angular speed is often measured in RPM, or revolutions per minute. In this paper, we will use S.I. units for angular speed: radians per second. There are about six radians in one revolution. A circle can be divided into 360 degrees, but it can also be divided into \(2\pi\) radians. \(\pi\) is equal to 3.14159..., so \(2\pi\) is equal to 6.28318... Therefore, one radian is approximately equal to one-sixth of a circle, or about 60 degrees. Radians are a little bit weird, but they have one very useful property. If one cuts a piece of pie in such a way that the length of the circumference of the piece is equal to the radius, then the subtended angle at the center of the piece is exactly equal to one radian. In other words, radians are very useful when it comes to relating angles around a circle to their corresponding arc lengths.

For the purposes of continuing with this example, considering the rear wheels only, let's define \(M_{\text{rearwheels}}\) as the mass of the rear wheels and \(I_{\text{rearwheels}}\) as the moment of inertia of the rear wheels. We can then re-write Equations (9) and (10) in terms of linear and angular accelerations as follows:

\[
\begin{align*}
\text{Horizontal Acceleration} & = \left( \frac{F_{g-w} - F_{t-a}}{M_{\text{rearwheels}}} \right) \\
\text{Clockwise Acceleration} & = \left( \frac{\tau_{\text{twr}} - RF_{g-w}}{I_{\text{rearwheels}}} \right)
\end{align*}
\] (11)

We can expand the \textit{Horizontal Acceleration} in terms of the distance by which the rear wheels, and the whole rig, have travelled down the course. Let's use the symbol \(D\) for the horizontal position of the rig at any instant of time. We can select any physical point along the side of the rig to use as a datum line to which we measure the distance \(D\). We can do this so long as the sled remains attached to the truck and
the tow chain remains straight, since everything then moves along in tandem. Suppose we use the hitch on the sled as the datum line. We will define \( D = 0 \) as the location of the sled's hitch at the start of the run. Once the sled begins to move, the distance \( D \) will be the distance from the hitch's starting position to its current position at any given time.

Let's use the symbol \( S \) for the instantaneous speed of the rig. Since the speed \( S \) is the rate at which the horizontal distance \( D \) increases, we can relate them using the following time-derivative:

\[
S = \frac{dD}{dt} \quad (12)
\]

In a similar way, the \textit{Horizontal Acceleration} is equal to the rate of change of the speed, so that:

\[
\text{Horizontal Acceleration} = \frac{dS}{dt} = \frac{d^2D}{dt^2} \quad (13)
\]

Similarly, we can expand \textit{Clockwise Acceleration} in terms of the angle by which the rear wheels have turned since the start of the run. We need some way to keep track of the angle by which the rear wheels have rotated. Let's use the symbol \( \phi \), which is the Greek letter "phi", for the cumulative angle of rotation. We can, for example, think of a chalk line drawn on the outside of the rear wheels from the center of the axle to the tread. Let's draw this line just before the start of the run so that it points straight downwards. Once the rig starts to move, the line will start to rotate. Angle \( \phi \) will start at \( 0^\circ \) and then grow as the wheels turn. Let's also define the symbol \( \omega \), which is the Greek small letter "omega", as the angular speed at which the rear wheels are turning at any instant of time. Then, the instantaneous angular speed is given by the time-derivative of the angle:

\[
\omega = \frac{d\phi}{dt} \quad (14)
\]

and the instantaneous angular acceleration is given by:

\[
\text{Clockwise Acceleration} = \frac{d\omega}{dt} = \frac{d^2\phi}{dt^2} \quad (15)
\]

The accelerations in Equation (11) can now be expressed directly in terms of distance and angle as:

\[
\begin{align*}
\frac{d^2D}{dt^2} &= \frac{F_g - F_t}{M_{\text{rear wheels}}} \\
\frac{d^2\phi}{dt^2} &= \frac{\tau_{\text{twr}} - RF_g}{l_{\text{rear wheels}}}
\end{align*}
\quad (16)
\]

Let me reiterate the assumption for our simplified model, that the rear wheels do not slip. If the rear wheels do not slip, then there is always a simple and direct relationship between the distance \( D \) the rig has travelled and the angle \( \phi \) the rear wheels have turned. To see the relationship, consider the following figure. It shows the rear wheels at the start of the run and then, separately, a short time later after the rig has started moving. The run starts at time \( t = 0 \), at which time both distance \( D \) and angle \( \phi \) are zero. A short time later, at time \( t = \Delta t \), say, the distance travelled is \( \Delta D \) and the angle turned is \( \Delta \phi \).
Since the wheels did not slip, that part of the circumference defined by angle $\Delta \varphi$ must be exactly equal to the distance $\Delta D$ travelled by the axle. If angle $\varphi$ is measured in radians, we can use the beautiful relationship that:

$$\Delta D = R \Delta \varphi \quad (17)$$

where $R$ is the radius of the wheels. (As an easy reality check, imagine the first full revolution of the wheels. Angle $\Delta \varphi$ is a complete circle, or $2\pi$ radians. This makes $\Delta D$ equal to $2\pi r$, which happens to be the circumference of the wheels.) Since Equation (17) is valid for any arbitrary time $\Delta t$ after the start of a run, it must hold at all times during a run (or, at least, up to the point where the wheels might start to slip). In general, then:

$$D = R \varphi \quad (18)$$

Since differentiation is a linear operator and the radius $R$ is constant, it follows that:

$$\frac{d^2 D}{dt^2} = R \frac{d^2 \varphi}{dt^2} \quad (19)$$

We can therefore combine the two expressions in Equation (16) as follows:

$$\frac{F_{g \rightarrow w} - F_{t \rightarrow a}}{M_{\text{rearwheels}}} = R \frac{\tau_{\text{twr}} - RF_{g \rightarrow w}}{l_{\text{rearwheels}}} \quad (20)$$

We can sort this out to isolate the force of the truck-on-the-axle. We get:

$$F_{t \rightarrow a} = F_{g \rightarrow w} - R \frac{M_{\text{rearwheels}}}{l_{\text{rearwheels}}} (\tau_{\text{twr}} - RF_{g \rightarrow w}) \quad (21)$$

We have already looked at the force of the ground-on-the-wheels. The expression is Equation (7). Substituting gives:

$$F_{t \rightarrow a} = \frac{\tau_{\text{twr}}}{R} - R \frac{M_{\text{rearwheels}}}{l_{\text{rearwheels}}} \left(\tau_{\text{twr}} - RF_{g \rightarrow w} \frac{\tau_{\text{twr}}}{R}\right) = \frac{\tau_{\text{twr}}}{R} = F_{g \rightarrow w} \quad (22)$$
Since the axle pushes as hard on the truck's chassis \( (F_{a\rightarrow t}) \) as the chassis pushes on the axle \( (F_{t\rightarrow a}) \), it follows that the force of the ground on the bottom of the rear wheels is transmitted through the wheels into the force which accelerates the truck and, hence, the whole rig.

Some readers may say that this is obvious. So, it is just as well that the physics confirms it.

**Part III -- The dynamics of the sled**

I repeat in the following figure the information we have about the sled. The figure is a "free body diagram" of the sled. All of the forces acting on the sled are shown at the points where they act. It is free to move in any direction, or to rotate if need be. All of the physical constraints which could prevent the sled from translating or rotating, like the ground, have been represented by the forces shown. In essence, one could think of the sled as floating somewhere in outer space. The question is: how does the sled respond to these forces? The study of the response of a body like the sled to forces and torques is called dynamics.

![Free Body Diagram of the Sled](image)

I have also shown at the right in the figure a pair of arrows which we can use to keep track of distances and speeds in two directions. It is customary to label the horizontal direction, which is parallel to the ground surface in our model, as the \( \hat{x} \)-direction. (The little caret over the \( x \) denotes that this quantity is a direction.) At right angles to the \( \hat{x} \)-direction is the \( \hat{y} \)-direction. In our model, it points straight upwards, directly opposite to the direction in which gravity pulls. Only two directions are needed to describe motion in the plane of the page. To describe torques, it is convenient to add a third direction, pointing straight out of the page. We will call this third direction the \( \hat{z} \)-direction. The figure to the right shows the three axes.

Everything of interest in our simplified model takes place in the \( \hat{x}-\hat{y} \) plane. But the third dimension \( \hat{z} \) is useful because it helps us keep track of torque and consequent rotations. If an off-center force and its associated lever-arm are such that it tends to rotate the sled in the direction shown in the figure, then we will call the torque "positive". Note that this direction is counterclockwise when seen in the \( \hat{x}-\hat{y} \) plane (and happens to be opposite to the direction in which the wheels turn).

To measure the torques exerted on the sled, and its tendency to rotate, we also need to select an axis of rotation. Selecting such an axis of rotation is not to say that the sled will rotate about this axis -- it will not rotate at all. It is just that we need to measure all the lever-arms from some common point. I have
arbitrarily chosen to use the center of the rear axle as the proposed rotation axis and will measure all the lever-arms from there.

Let me explicitly state one more assumption we will rely on in the simplified model. We will ignore the energy absorbed by the rotation of the wheels. The wheels have mass. It takes a force to move them. We won’t ignore that. But, the wheels also have moments of inertia and it takes additional power, or energy, to cause them to turn. It is this additional energy which we will ignore for the time being. In a subsequent paper, we will look at things like that.

To see what happens to the sled, let’s add up all the forces in the $\hat{x}$ and $\hat{y}$-directions and all the torques around the center of the rear axle in the $\hat{z}$-direction.

\[
\sum F_x = T - D_{sf} \tag{23A}
\]
\[
\sum F_y = F_{sf} + F_{sr} - M_{sf}g - M_{sm}g - M_{sr}g \tag{23B}
\]
\[
\sum \tau_z = F_{sf}W_s - M_{sf}gW_s - M_{sm}gD_w - D_{sf}H_s \tag{23C}
\]

Note that three of the forces -- $T$, $F_{sr}$ and $M_{sr}g$ -- do not generate torques because their lines-of-action pass through the rear axle.

The sled does not accelerate upwards or downwards to any meaningful or sustained degree, so the sum of the forces in the $\hat{y}$-direction must be zero. This means that the right-hand side of Equation (23B) must be zero. Nor does the sled rotate to any meaningful or sustained degree. The pan will dig into the ground a little bit but, compared with the length of the sled, the amount of rotation is negligible. The sum of the torques, and the right-hand side of Equation (23C), will also be zero.

But, the sled does accelerate in the $\hat{x}$-direction. The sum of the forces in the $\hat{x}$-direction will not be zero. Here, we can apply Newton’s Law for dynamics one more time. The total mass of the sled is the sum of three of the masses we defined in Part I: $M_{sf}$, $M_{sm}$, and $M_{sr}$. If we let the symbol $A_s$ be the horizontal acceleration of the sled, then Newton’s Law for acceleration in that direction can be written as:

\[
\sum F_x = (M_{sf} + M_{sm} + M_{sr})A_s \tag{24}
\]

The three equations of motion can then be re-stated as follows:

\[
(M_{sf} + M_{sm} + M_{sr})A_s = T - D_{sf} \tag{25A}
\]
\[
F_{sf} + F_{sr} - (M_{sf} + M_{sm} + M_{sr})g = 0 \tag{25B}
\]
\[
F_{sf}W_s - (M_{sf}W_s + M_{sm}D_w)g - D_{sf}H_s = 0 \tag{25C}
\]

There is a fundamental reason why there happen to be exactly three equations of motion. A rigid body confined to a plane can translate in two distinct directions and rotate in only one. In a three-dimensional model, on the other hand, a rigid body can translate in three distinct directions and also rotate in three distinct directions, so there would be a total of six equations of motions.

With three equations at our disposal, we should be able to calculate three things. So, let’s inspect the three equations to sort out which quantities we know and which we don’t. From the specifications of the sled, we know the masses $M_{sf}$, $M_{sm}$ and $M_{sr}$ and the dimensions $W_s$ and $D_w$. Although the moveable
weight’s location $D_w$ is not a constant, we can calculate where it is at any given time, so it is not "unknown" in the usual sense. We also know, or can calculate using Equation (2), the drag of the pan $D_{sf}$ from the other quantities, so it is not really an unknown either. Finally, we know the gravitational acceleration $g$. The quantities we do not know are: the acceleration $A_s$, the tow chain tension $T$ and the two upward forces exerted by the ground $F_{sf}$ and $F_{sr}$.

We have four unknown quantities but only three equations, so we will not be able to get a complete solution. That makes sense, because we have not yet taken the truck into account. When we look at the dynamics of the truck, we will get that extra bit of information, and will have our complete solution.

In the meantime, though, we can start to combine the expressions we do have. First, let’s insert the expressions we already have for $D_w$, using Equation (1), and for $D_{sf}$, using Equation (2). We get:

$$\begin{align*}
(M_{sf} + M_{sm} + M_{sr})A_s &= T - C_{pd}F_{sf} \quad (26A) \\
F_{sf} + F_{sr} - (M_{sf} + M_{sm} + M_{sr})g &= 0 \quad (26B) \\
F_{sf}(W_s - C_{pd}H_s) - [M_{sf}W_s + M_{sm}(D_{w0} + R_w t)]g &= 0 \quad (26C)
\end{align*}$$

Here is what we will do next. We can re-arrange the last equation to get an expression for $F_{sf}$. We can substitute that into the middle equation to get an expression for $F_{sr}$. And, we can also substitute the expression for $F_{sf}$ into the first equation to get the relationship between the acceleration $A_s$ and the tow chain tension $T$. The procedure we will use to calculate the unknown quantities is this:

$$\begin{align*}
F_{sf} &= \frac{[M_{sf}W_s + M_{sm}(D_{w0} + R_w t)]g}{W_s - C_{pd}H_s} \quad (27 - 1st) \\
F_{sr} &= (M_{sf} + M_{sm} + M_{sr})g - F_{sf} \quad (27 - 2nd) \\
A_s &= \frac{T - C_{pd}F_{sf}}{M_{sf} + M_{sm} + M_{sr}} \quad (27 - 3rd)
\end{align*}$$

**Part IV -- The dynamics of the truck**

The following figure is the free body diagram of the truck.

Note that the torque $\tau_{t_w}$ exerted by the chassis on the rear wheels is nowhere to be seen in this diagram. In this diagram, we are treating the whole truck as a rigid body. Being rigid, it has no moving parts. The torque which the chassis exerts on the rear wheels, and the counteracting torque which the rear wheels exert on the chassis, cancel each other out. They are internal stresses of the truck, which have no direct effect on its overall motion. Instead, the entire effect of the ground on the truck, which is the ultimate result of the powertrain, is the force shown in purple. This is the force of the ground-on-the-wheels $F_{g-w}$.
which we examined in the section before last. As we did for the sled, we will ignore in the simplified model the additional energy needed to rotate the wheels.

We will use the same $\hat{x}$-$\hat{y}$-$\hat{z}$ co-ordinate frame of reference as we did for the sled. This time, though, we will position the frame of reference at the center of the truck's rear axle. Torques will be calculated with their lever-arms measured from the center of the truck's rear axle.

As we did for the sled, we will write down the sums of the forces and the torques. They are:

\[
\sum F_x = F_{g\rightarrow w} - T \quad (28A)
\]
\[
\sum F_y = F_{tf} + F_{tr} - M_{tr}g - M_{tr}g \quad (28B)
\]
\[
\sum \tau_z = F_{tf} W_t - M_{tf} gw_t + F_{g\rightarrow w} R \quad (28C)
\]

Three of the forces -- $T$, $F_{tr}$ and $M_{tr}g$ -- do not generate torques because their lines-of-action pass through the rear axle.

In the last equation, I have used the wheels' radius $R$ as the lever-arm for the torque induced by the force of the ground-on-the-wheels. In the simplified model, the wheels' radius is that same as the deck height $H_s$ of the sled.

Like the sled, the truck does not accelerate in the vertical direction nor, if we ignore the possible event of a wheelie, does it rotate. Therefore, the left-hand sides of the latter two equations will be zero. To apply Newton's Law to the horizontal, or $\hat{x}$-axis, acceleration, let's use the symbol $A_t$ for the horizontal acceleration of the truck. Also recall that the total mass of the truck is $M_{tf} + M_{tr}$. Putting all this information into Equation (28) gives the following equations:

\[
(M_{tf} + M_{tr}) A_t = F_{g\rightarrow w} - T \quad (29A)
\]
\[
F_{tf} + F_{tr} - (M_{tf} + M_{tr}) g = 0 \quad (29B)
\]
\[
F_{tf} W_t - M_{tf} gw_t + F_{g\rightarrow w} R = 0 \quad (29C)
\]

We have already seen that the ground-on-the-wheels $F_{g\rightarrow w}$ is given by $\tau_{twr}/R$ or, alternatively, by $\tau_{twr}/H_s$ since the wheels' radius is assumed to be the same as the deck height. Substituting this relationship gives:

\[
(M_{tf} + M_{tr}) A_t = \tau_{twr}/H_s - T \quad (30A)
\]
\[
F_{tf} + F_{tr} - (M_{tf} + M_{tr}) g = 0 \quad (30B)
\]
\[
F_{tf} W_t - M_{tf} gw_t + \tau_{twr} = 0 \quad (30C)
\]

Once again, we have three equations of motion. Four quantities are unknown: the acceleration of the truck $A_t$, the tow chain tension $T$ and the two upward forces exerted by the ground $F_{tf}$ and $F_{tr}$.

As we did for the sled, we will solve the three equations in reverse order. We will find $F_{tf}$ using the last equation and then find $F_{tr}$ using the middle equation. The remaining equation will be the relationship between the truck's acceleration $A_t$ and the tow chain tension $T$. 

~ 17 ~
The procedure is as follows:

\[ F_{tf} = \frac{M_{tf}gW_t - \tau_{twr}}{W_t} \quad (31 - 1st) \]
\[ F_{tr} = (M_{tf} + M_{tr})g - F_{tf} \quad (31 - 2nd) \]
\[ A_t = \frac{\tau_{twr} - T}{M_{tf} + M_{tr}} \quad (31 - 3rd) \]

**Part V -- The dynamics of the rig**

Let's compare the expressions we have for the accelerations of the sled, in Equation (27 - 3rd), and for the truck, in Equation (31 - 3rd). As long as the tow chain stays taut, they must be equal. The sled will accelerate at the same rate as the truck. If the sled accelerated at a different rate than the truck, then it would get closer to or further away from the truck. So long as the accelerations are equal, we get:

\[ A_t = A_s \]
\[ \rightarrow \frac{\tau_{twr} - T}{M_{tf} + M_{tr}} = \frac{T - C_{pd}F_{sf}}{M_{sf} + M_{sm} + M_{sr}} \quad (32) \]

This equation involves only one unknown, the tension \( T \) in the tow chain. We can solve for the tension by rearranging Equation (32) through the following steps:

\[ (M_{sf} + M_{sm} + M_{sr}) \left( \frac{\tau_{twr} - T}{H_s} \right) = (M_{tf} + M_{tr}) \left( T - C_{pd}F_{sf} \right) \]
\[ \rightarrow (M_{tf} + M_{tr})(T - C_{pd}F_{sf}) + (M_{sf} + M_{sm} + M_{sr}) \left( T - \frac{\tau}{H_s} \right) = 0 \]
\[ \rightarrow (M_{tf} + M_{tr} + M_{sf} + M_{sm} + M_{sr})T = (M_{tf} + M_{tr})C_{pd}F_{sf} + (M_{sf} + M_{sm} + M_{sr}) \frac{\tau}{H_s} \]
\[ \rightarrow T = \frac{(M_{tf} + M_{tr})C_{pd}F_{sf} + (M_{sf} + M_{sm} + M_{sr}) \frac{\tau}{H_s}}{M_{tf} + M_{tr} + M_{sf} + M_{sm} + M_{sr}} \quad (33) \]

The sums of the individual masses are arranged in convenient groups. Looking at the first subscripts, it is clear that we are dealing with the total mass of the truck, the total mass of the sled and, in the denominator, the total mass of the rig. Defining the obvious symbols for these groups of masses, we can write Equation (33) as:

\[ T = \frac{M_{truck}C_{pd}F_{sf} + M_{sled} \frac{\tau_{twr}}{H_s}}{M_{truck} + M_{sled}} \quad (34) \]

The force of the ground on the pan \( F_{sf} \) is given by Equation (27 - 1st) in more fundamental terms. Substituting for \( F_{sf} \) in Equation (34) gives:

\[ T = \frac{M_{truck}C_{pd} \left[ \frac{M_{sf}W_s + M_{sm}(D_{wo} + R_w \ell)}{W_s - C_{pd}H_s} \right] g + M_{sled} \frac{\tau_{twr}}{H_s}}{M_{truck} + M_{sled}} \quad (35) \]
So, we now have a complete and self-contained expression for the tension in the tow chain. We can use it to calculate the acceleration of the rig using either Equation (25 – 3rd) to calculate \( A_s \) or Equation (31 – 3rd) to calculate \( A_t \). Either way will give the same result. We will use the latter. We might as well use the symbol \( A \), with no subscript, for the acceleration of the rig. Substituting \( T \) into Equation (31 – 3rd) gives:

\[
A = \frac{\frac{\tau_{\text{towr}}}{H_s} - \frac{M_{\text{truck}} C_{pd} \left( \frac{M_s W_s + M_{sm} (D_{w0} + R_w t)}{W_s - C_{pd} H_s} \right) g}{M_{\text{truck}} + M_{\text{sled}}}}{M_{\text{truck}}}
\]

(36)

**Part VI -- The kinematics of the rig**

As I mentioned above, "dynamics" is the study of an object's response to forces and torques. The object's response is normally expressed in terms of its accelerations in various dimensions. "Kinematics" on the other hand deals with the relationship between accelerations and the distances and speeds which the accelerations cause. Since we know the horizontal acceleration of the rig, we can use it to calculate the horizontal speed and then the horizontal distance travelled. As above, we will use the symbol \( S \) for the instantaneous speed of the rig and \( D \) for its instantaneous position. We will assume that the run start at time \( t = 0 \), with the rig stationary \( (S = 0) \) at the starting line \( (D = 0) \).

The acceleration is the rate of change of the speed. Stated inversely, the speed is the cumulative product of acceleration and time. The following figure may be helpful in understanding the relationship.

![Speed and Acceleration Graphs](image)

The two graphs show the progress of an object, such as our rig, as time changes. The black curve in the left-hand graph is the speed of the rig. At time \( t = 0 \), the rig is stationary so its speed is also zero, \( S = 0 \). As time passes, it moves to positions further out along the "Time t" axis. One can see that the rig has a positive speed after time \( t = 0 \). Not only that, but the rig is moving faster and faster as time passes. In other words, it is accelerating. Let's look at a particular instant of time, at time \( t \), which is identified by the dotted black line. The speed of the rig at that instant is shown by the heavy black dot. The orange line has been drawn so that it is exactly parallel to the curve where it passes through the black dot. Mathematicians would say that the orange line is "tangent to" the curve at this point. We are interested in the slope of the orange line. The slope of the line is a measure of how quickly its vertical value increases for a given horizontal change. I have shown a pair of vertical and horizontal changes in the neighbourhood of the black dot. The dotted green line is the change in the vertical value, called the "rise", which is associated with the change in the horizontal value, called the "run", shown by the dotted red line. The slope of the orange line is their ratio:
The length of the dotted green line represents a certain change in speed. The length of the dotted red line represents a certain change in time. The ratio, of the change in speed divided by the change in time, is the very definition of acceleration. In other words, the acceleration of the rig at any instant of time is the slope of the speed-verse-time graph at that instant.

Now, let's look at the right-hand graph in the figure above. The black curve in the right-hand graph is the acceleration of the rig. The acceleration of the rig is not zero at time \( t = 0 \). Even though the rig is stationary, so that its speed is zero, it can still be accelerating. The initial acceleration will be determined by Newton's Law. Whatever force the engine can impart in first gear from a standing start, divided by the mass of the rig, will be (approximately) the initial acceleration. I have drawn the curve in such a way that the acceleration increases without bound as time passes. That is unlikely to be the case in a real truck pull. The acceleration might increase for a short time but, once the moveable weight starts to load the pan, the rig will start to decelerate. Once again, we will look at a particular instant of time, time \( t \), which is again identified by a dotted black line. The speed of the rig at this instant is equal to the area under the acceleration-versus-time curve from time \( t = 0 \) up to this instant.

There is a mathematical equivalency between the area under the acceleration-versus-time curve and the slope of the speed-versus-time curve. The equivalency is most easily expressed using the Calculus, as follows:

\[
\begin{align*}
A &= \frac{dS}{dt} \quad \text{for } t = t \\
S &= \int_{t=0}^{t} A dt
\end{align*}
\]  

(38)

In the first expression, the acceleration is equal to the time-derivative of the speed. It is the limiting case of Equation (37), in which the red and green dotted lines are drawn at points which are infinitely close to the black dot. Similarly, the time-integral in the second expression is the way mathematicians add up the area under the \( A \)-versus-\( t \) curve.

Look back at Equation (36), which is our expression for the acceleration of the rig versus time. In that expression, time \( t \) appears explicitly in connection with the term which describes the location of the moveable weight. Everything else in Equation (36) is known to be constant except for one thing: the torque \( \tau_{\text{twr}} \) delivered to the rear wheels of the truck by the powertrain. This torque depends on a lot of factors but is unlikely to be constant throughout the duration of a complete run. Many of these factors depend on variables other than time, such as gear selection, fuel flow rate, engine design, and so on. Even if we knew all of these dependencies were, it is unlikely that we could ever combine them into an expression for the torque which depended only on time and not on anything else. In other words, a closed-form integration of Equation (36), which would give us the speed of the rig, would be a crude approximation only. (But can still provide some very useful information, as we will see below.)

In any event, let's continue with the kinematics to determine how we can derive the distance travelled from the speed. It turns out that the relationship between distance and speed is virtually identical to the relationship between speed and acceleration. The following figure shows the former.
The instantaneous speed of the rig is the slope of the distance-versus-time curve. The equivalent to Equation (37) is:

\[
\text{Slope of orange line} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{Change in distance}}{\text{Change in time}} \tag{39}
\]

And, the distance travelled at any instant of time is equal to the area under the speed-versus-time curve from time \( t = 0 \) up to that instant. Really, the only difference between the relationships is that the rules of the truck pull require that the initial speed be zero but allow the initial acceleration to be non-zero.

The mathematical relationships between speed and distance are the following:

\[
\begin{align*}
S &= \frac{dD}{dt} \bigg|_{t=t} \\
D &= \int_{t=0}^{t} S \, dt 
\end{align*} \tag{40}
\]

If we knew how the torque changes with time, we could use the second expression in Equation (38) to calculate the speed at any instant of time. We could then use the second expression in Equation (40) to calculate the distance travelled up to that instant.

Generally speaking, there are two ways in which we could proceed. We will explore both of them. In the next section of this paper, we will make the simplifying assumption that the powertrain torque is indeed constant throughout the run. We can then integrate in closed-form. In a subsequent paper, we will look at some of the factors which affect the torque. Based on those relationships, we will carry out the integrations "numerically", which is something I will explain at that time.

**Part VII -- Distance travelled when the crankshaft torque is constant**

In this part, we will assume that the powertrain delivers the same amount of torque to the rear wheels at all times. Just so there is no confusion about the point, I will use the symbol \( \tau_{\text{twr}}^{\text{const}} \) for the constant amount of torque applied to the drive wheels. The superscript has no special meaning, but will serve as a reminder that this value arises from a special assumption. For convenience, I will re-state Equation (36) here. The acceleration of the rig is:
Note that I have written the acceleration as $A(t)$ because its dependence on time is now shown explicitly and completely on the right-hand side. Everything on the right-hand side other than time $t$ is constant with respect to time. The algebra will be simplified if we collect factors which are constant. If we do that, we arrive at:

$$A(t) = Pt + Q \quad (42)$$

where $P$ and $Q$ are constants given by:

$$P = -\frac{C_{pd} \left( \frac{M_{sm} R_w g}{W_s - C_{pd} H_s} \right)}{M_{truck} + M_{sled}} \quad (43A)$$

$$Q = -\frac{\tau_{t_{fwr}} C_{pd} \left( \frac{M_{sfn} W_s + M_{sm} D_{w0} g}{W_s - C_{pd} H_s} \right) + M_{sled} \frac{\tau_{t_{fwr}}}{H_s}}{M_{truck}} \quad (43B)$$

The form of the acceleration in Equation (42) is easy to integrate. A first integration will give us the speed. We proceed as follows:

$$S(t) = \int_{t=0}^{t=t} A(t) dt = \int_{t=0}^{t=t} (Pt + Q) dt = P \int_{t=0}^{t=t} t dt + Q \int_{t=0}^{t=t} dt = P \left( \frac{1}{2} t^2 \right)_{t=0}^{t=t} + Q \left( t \right)_{t=0}^{t=t} = \frac{1}{2} Pt^2 + Qt \quad (44)$$

As a quick reality check, note that the right-hand side evaluates to zero when we substitute time $t = 0$. In other words, the expression for the speed satisfies the expected initial condition. Now, let's integrate the speed to get the distance travelled.

$$D(t) = \int_{t=0}^{t=t} S(t) dt$$
Substituting time \( t = 0 \) shows that the distance travelled is zero at the start of the run, so the expression for distance satisfies its initial condition as well.

\[ D(t) = \int_{t=0}^{t=t} \left( \frac{1}{2}pt^2 + Qt \right) dt \]

\[ = \frac{1}{2}p \int_{t=0}^{t=t} t^2 dt + Q \int_{t=0}^{t=t} t dt \]

\[ = \frac{1}{2}p \left( \frac{1}{3}t^3 \right)_{t=0}^{t=t} + Q \left( \frac{1}{2}t^2 \right)_{t=0}^{t=t} \]

\[ = \frac{1}{6}pt^3 + \frac{1}{2}Qt^2 \quad (45) \]

**Part VIII -- A numerical example with constant powertrain torque**

Let's consider a stock 1997 Ford F-150 SuperCab. The following paragraphs describe some of the manufacturer's specifications and the calculations I did to obtain values for certain of the parameters in Equation (41).

- The truck weighs 4,980 pounds. This specification is for a driverless vehicle with no fluids. Add a 200 pound driver and 100 pounds of fluids and the total weight becomes 5,280 pounds.

- The "basic weight distribution" is 61:39. The basic weight distribution gives the relative percentages of the truck's total weight on the front axle (the first number) and the rear axle (the second number). These relative percentages apply to the driverless, dry weight, of 4,980 pounds. The weight at the front axle is 61% of 4,980 pounds, or 3,040 pounds. The weight at the rear axle is 39% of 4,980 pounds, or 1,940 pounds. Let's assume that the placement of the driver and fluids is such that two-thirds of their collective weight, or 200 pounds, is borne by the front axle and the remaining 100 pounds is borne by the rear axle. All told, then, we have 3,240 pounds on the front axle and 2,040 pounds on the rear axle. Since we want to use S.I. units, we need to convert these masses into kilograms. One pound is equivalent to 454 grams, or 0.454 kilograms. Applying this scaling factor, we get: a concentrated mass at the front wheels of 1,470 kilograms and a concentrated mass at the rear wheels of 926 kilograms \((M_{tf} = 1,470 \text{ kg})\) and a concentrated mass at the rear wheels of 926 kilograms \((M_{tr} = 926 \text{ kg})\).

- The wheelbase of the truck is reported to be 138.8 inches. The S.I. units we will use are meters. Since there are 2.54 centimeters in an inch, the wheelbase is equal to \(138.8 \times 2.54 = 352.6\) centimeters. Divide by 100 to convert centimeters to meters and we conclude that the wheelbase is 3.53 meters \((W_e = 3.53 \text{ m})\). (Aside: There is no benefit to trying to be overly precise when using a rough mathematical model. Three significant digits is more than enough.)

- The stock truck comes equipped with 265/70R17 tires, which have a nominal diameter of \(31\frac{1}{2}\) inches. Their radius is one-half that, or \(15\frac{3}{4}\) inches. We will use the radius of the truck's tires as our estimate for the basic deck height of the rig. The radius is converted into meters by multiplying by 0.0254, so the deck height is \(15\frac{3}{4} \times 0.0254 = 0.400\) meters \((H_s = 0.400 \text{ m})\).

- The stock engine is said to develop 235 horsepower and 330 foot-pounds of torque. I will assume that we are running an unmodified engine. It will not be possible to run the course completely at
maximum torque. Let's assume that the best we can do is run at an average of 75% of peak torque, or 248 foot-pounds.

- Let's assume we run the entire course in first gear, for which the gear ratio is 2.84. The differential has a gear ratio of 3.73. As the rotational speed decreases down the drivetrain, the torque will be scaled up by corresponding ratios. The nominal torque on the rear wheels will be $248 \times 2.84 \times 3.73 = 2630$ foot-pounds. But, there are losses in both the transmission and differential. If each has an efficiency of 90%, then the actual torque on the rear wheels will be 81% of the nominal amount, or 2,130 foot-pounds.

- We need to convert foot-pounds into Newtons-meters. Let's convert from feet into meters first. A foot contains 12 inches, each of which contains 2.54 centimeters, each of which contains one-hundredth of a meter. The nominal torque can be re-stated as $2,130 \times 12 \times 2.54 \times 0.01 = 649$ meter-pounds. Next, we need to convert pounds into Newtons. (A couple of paragraphs above, we converted pounds of mass into kilograms. Here, we are going to convert pounds of force into Newtons. The old English system of units is not very refined, which is one of the benefits from using S.I. units) One Newton is pretty close to the weight of the patty in one of McDonald's Quarter-Pound hamburgers. I will not go through the background calculations to show why that is so, but will merely state that the exact conversion is one pound equals 4.448222 Newtons. All told, we set the torque in the equations to $649 \times 4.448222 = 2890$ Newton-meters ($\tau_{\text{const}} = 2890 \text{Nm}$).

That completes the specification of the parameters which describe the truck in our simplified model. Now, let's look at a sled. There is no such thing as a standard sled. The dimensions and weights of any sled you encounter will depend on where you pull, and possibly the class in which you compete. The specifications described below are a composite derived from different sleds which two wheel drive pickup trucks might be expected to pull. In my neck of the woods, this class is known as the Hillbilly class.

- The sled is about 25 feet long from end-to-end. It happens to have two rear axles, rather than just one, but that does not affect our simplified model. The rear axles are not located near the back end of the sled, where they would be on a highway trailer, but much further forward. The wheelbase, measured from the center of the rear axles to the center of the pan, is about 12 feet. The pan itself is quite large, so its center is quite a ways in from the front of the sled. In S.I. units, the wheelbase is equal to $12 \times 12 \times 2.54 \times 0.01 = 3.66$ meters ($W_{\text{f}} = 3.66 \text{ m}$). This distance might feel too short. On the sled I am imagining, however, the moveable weight starts a couple of feet behind the rear wheels and moves forward to a point at the front of the pan. The moveable weight travels more than 12 feet; it travels about 19 feet.

- Before a run, the moveable weight is positioned to a point which is about 2 feet behind the center of the rear axles. This is the starting distance $D_{w0}$ we used above when calculating how the location of the moveable weight changes with time. In S.I. units, this is equal to $-2 \times 12 \times 2.54 \times 0.01 = -0.610$ meters ($D_{w0} = -0.610 \text{ m}$). Remember that we defined the distance $D_w$ as the distance of the moveable weight in front of the rear wheels, so we use a minus sign for locations which are aft of the rear wheels.

- During the course of a run, the moveable weight moves forward from its starting position at a constant rate. When the sled is pulled by large trucks and tractors, the moveable weight moves the total distance of 19 feet in about 10 seconds. This represents a speed of 1.9 feet per second. This is the rate $R_w$ we used above when calculating how the location of the moveable weight changes with time. The sled's operator can control the rate. For smaller trucks and tractors, he
reduces the speed at which the mass moves forward, since they would otherwise stall too early in the run. For the base case in our simplified model, we will assume that the moveable weight moves the full 19 feet in 20 seconds, half the rate used for the big rigs. In S.I. units, this is equal to $0.95 \times 12 \times 2.54 \times 0.01 = 0.290$ meters per second ($R_w = 0.290 \text{ m/s}$).

- The moveable weight used for the big rigs weighs 6,500 pounds or even more. Some of this weight is removed from the sled when light rigs are run. For the base case in this paper, I will use 4,000 pounds. We can convert these pounds of mass to kilograms by multiplying by 0.454, as we did above. We get a mass of $4,000 \times 0.454 = 1,820$ kilograms ($M_{sm} = 1820 \text{ kg}$).

- Now, let's consider the sled with the moveable weight removed. The empty sled has some similarities to the kind of flatbed trailer one sees in a tractor-trailer rig running down the highway carrying a load of steel pipe or coils. It is even more like the chassis used to dray 40-foot international shipping containers. But it has a lighter construction than either. Container chasses, for example, are built to carry containers which weigh up to 50,000 pounds. The moveable weight weighs only a fraction of that. I estimate the the empty weight of the sled to be about 4,200 pounds, or $4,200 \times 0.454 = 1,910$ kilograms ($M_{sf} + M_{sr} = 1,910 \text{ kg}$). Guessing the basic weight distribution is a little trickier. The operator's shack at the rear end and the forward location of the rear axles are intended to put most of the empty weight on the rear wheels. There are placed where they are specifically to reduce the starting weight on the pan. The fact that the moveable weight starts off at a location behind the rear wheels further reduces the starting load on the pan. I would guess that the empty weight on the pan is 1,500 pounds and the empty weight on the rear wheels is 2,700 pounds ($M_{sf} = 680 \text{ kg}$ and $M_{sr} = 1,230 \text{ kg}$).

- One can look up the gravitational acceleration $g$ in most Physics textbooks or on many websites. It is usually given in S.I units as 9.81 meters per second squared ($g = 9.81 \text{ m/s}^2$), but some old-timers may remember it as 32.2 feet per second per second.

With one exception, that completes the specification of the parameters we need for the sled. The quantity we have not yet specified is the coefficient of dynamic friction which applies to the loaded pan as it is dragged across the ground. This is a parameter which must be determined by experiment. I looked at various sites on the internet, and observed the following:

- Tables on www.supercivilcd.com state that: (i) the coefficient of sliding friction of steel on steel is 0.30 and (ii) the coefficient of sliding friction of steel on concrete is 0.45. This site also reports that the friction is greater on dry surfaces than when the corresponding surfaces are wet.

- Comments posted on www.engineerboards.com include: (i) "NAVFAC gives a value of 0.4 for clean steel on gravel" and (ii) "0.3 would be conservative." (Aside: NAVFAC is the U.S. Naval Facilities Engineering Command. I didn't know the Navy did truck pulls.)

- The site www.finesoftware.eu includes a table of NAVFAC results, from which I extracted the following friction coefficients for "steel sheet piles against the following soils."

<table>
<thead>
<tr>
<th>Soil</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clean gravel, gravel-sand mixtures, well-graded</td>
<td>0.40</td>
</tr>
<tr>
<td>rock fill with spalls</td>
<td></td>
</tr>
<tr>
<td>Clean sand, silty sand-gravel mixture, single</td>
<td>0.30</td>
</tr>
<tr>
<td>size hard rock fill</td>
<td></td>
</tr>
<tr>
<td>Silty sand, gravel or sand mixed with silt or</td>
<td>0.25</td>
</tr>
<tr>
<td>clay</td>
<td></td>
</tr>
<tr>
<td>Fine sandy silt, nonplastic silt</td>
<td>0.20</td>
</tr>
</tbody>
</table>
Looking at this last table, it seems to me that the coefficient of dynamic friction is correlated to the size of the particles beneath the pan. Gravel has the greatest coefficient; silt has the smallest coefficient and sand is in the middle. This makes sense. The smaller the particles in the dirt are, the more easily they roll with respect to one another and the less resistance they offer to the loaded steel pan passing overhead. We would therefore expect that, when there are no free particles at all, like a concrete surface, the coefficient of sliding friction would be even greater. Indeed, that is exactly what we see reported on the first website mentioned above, where the steel-concrete coefficient is estimated to be 0.45. That wet soils have a lower coefficient of drag also makes sense. Any water in the soil acts like a lubricant, making it easier for the individual particles to rotate and slide past one another.

My conclusions are these:
- On a dry dirt track, the coefficient of friction will likely be in the range 0.30 to 0.40 and
- On a rainy day, the coefficient will be a little less, say, in the range 0.25 to 0.35.

For our numerical example, I will use the value $C_{pd} = 0.40$.

It is a straight-forward exercise to substitute these quantities into the expressions for constants $P$ and $Q$ in Equation (43). In the following steps, I have kept track of the units, just to make sure that everything is consistent.

\[
P = -\frac{C_{pd} \left( \frac{M_{sm} R_w g}{W_s - C_{pd} H_s} \right)}{M_{truck} + M_{sled}}
= -\frac{0.40 \left( \frac{1820 \text{ kg} \times 0.290 \text{ m/s} \times 9.81 \text{ m/s}^2}{3.66 \text{ m} - 0.40 \times 0.40 \text{ m}} \right)}{2396 \text{ kg} + 3730 \text{ kg}}
= -\frac{0.40 \left( \frac{5180 \text{ kg/m}^2/\text{s}^2}{3.50 \text{ m}} \right)}{6130 \text{ kg}}
= -0.0966 \text{ m/s}^3 \quad (46)
\]

The expression for $Q$ in Equation (43B) is convoluted enough that I will attack it in two steps. I will calculate the factor in square brackets in the numerator first.

\[
\left[ \frac{(M_{sf} W_s + M_{sm} D_{w0}) g}{W_s - C_{pd} H_s} \right] = \frac{(680 \text{ kg} \times 3.66 \text{ m} + 1820 \text{ kg} \times -0.610 \text{ m}) \times 9.81 \text{ m/s}^2}{3.66 \text{ m} - 0.40 \times 0.40 \text{ m}}
= \frac{13,520 \text{ kg/m}^2/\text{s}^2}{3.50 \text{ m}}
= 3860 \text{ kg/m/s}^2 \quad (47A)
\]

and then complete the calculation of $Q$ as follows:

\[
Q = -\frac{\frac{\tau^\text{const}}{H_s}}{M_{truck}} \left( \frac{M_{truck} C_{pd} \left[ \ldots \right] + M_{sled} \frac{\tau^\text{const}}{H_s}}{M_{truck} + M_{sled}} \right)
\]

\[\sim 26\]
When rationalizing units in the preceding steps, I used the relationship that one Newt

\[ \text{on} \] is equal to one \( \text{kilogram multiplied by one meter and divided by one second squared.} \] This relationship is easy to remember if one just recalls Newton's Law: force equals mass times acceleration. The unit of force is a Newton; the unit of mass is a kilo
gram and the unit of acceleration is meters per second squared.

We can now substitute the values for \( P \) and \( Q \) into the expression for the distance travelled by the rig which is given in Equation (45). Remember that time \( t \) is measured in seconds. Then:

\[
D(t) = \frac{1}{6} Pt^3 + \frac{1}{2} Qt^2
\]

\[
= \frac{1}{6} \times 0.0966 \, \text{m/s}^3 \times t^3 + \frac{1}{2} \times 0.931 \, \text{m/s}^2 \times t^2
\]

\[
= -0.0161 t^3 + 0.466t^2 \text{ meters}
\]

\[
= -0.0528 t^3 + 1.53t^2 \text{ feet} \quad (48)
\]

The following figure is a graph of this function, showing the distance travelled as a function of time.
The rig travels a maximum distance of just over 190 feet, and takes about 19 seconds to get there. That is indicated as point B in the graph. Apparently, it then starts to move back towards the starting position. What gives?

No, there is nothing wrong with Equation (48). The curve shown is a combination of: (i) a quadratic term, the $1.53t^2$, which has a positive coefficient and therefore looks like a concave-upwards parabola, and (ii) a cubic term, the $-0.0528t^3$, which has a negative coefficient and therefore overwhelms the parabola once time $t$ gets large enough, and forces the curve downwards.

What must be "wrong" is that, at some point during the run, one or more of the forces changed in such a way that the assumptions which underpin our simplified model are no longer met. We will see in a minute exactly what happens when we reach the envelope of the simplified model. It helps, first, to look at corresponding curves for the speed and acceleration over the same period of time.
In the three graphs above, I have identified certain points as A, B and C. Let me explain what happens at these three times.

At time A, which occurs about 9.8 second into the run, the acceleration has decreased to zero. Before this time, the truck had sufficient torque, not just to pull the sled forwards, but to accelerate as it was doing so. The moveable weight moves forward from the start of the run, and it is at this moment that its loading on the pan is such that the frictional drag the pan exerts is just equal to the force which the ground exerts on the bottom of the rear wheels of the truck.

Time A is also the moment when the rig reaches its maximum speed. The maximum speed it reaches is 14.8 feet per second. Let me convert this into miles per hour. There are 5,280 feet in one mile and 3,600 seconds in one hour, so the conversion is $14.8 \times 3600 / 5280 = 10.1$ miles per hour. That's not very fast. More importantly, the crankshaft speed which corresponds to this translational speed is well below the RPM at which the engine develops maximum horsepower. Tires with a diameter of 31½ inches have a circumference of 99.0 inches, or 8.25 feet. When the move forward at 14.8 feet per second, they turn 1.79 revolutions per second, or 108 RPM. Scaling back up the powertrain through the differential's 3.73 gear ratio and the transmission's 2.84 gear ratio (in first gear), the engine is running at 1140 RPM. Remember, too, this is the rig's maximum speed!

Between times A and B, the rig continues to move forward, but it is slowing down. At time B, it comes to a stop. The speed reaches zero and the rig has travelled down the course as far at it will go. Time B occurs 19.3 seconds into the run.

Physically, the truck might spin its wheels at this point. Or stall. But it will not pull the sled any further. On the other hand, Equation (48) continues to "run" and has the rig begin to move backwards. Let me explain why. The fault arises from Equation (2), which expresses the drag on the bottom of the pan as a constant (the coefficient of dynamic friction $C_{pd}$) multiplied by the vertical force between the pan and the ground ($F_{pd}$). This relationship is valid only so long as the pan is moving. Two things happen once the pan comes to a stop. The first is that the coefficient of sliding friction ceases to apply. Once the sled has stopped, we should be using the coefficient of static friction, which is a little bit greater. The second thing is actually more important than the slight increase in the coefficient of friction. The whole nature, or concept, of the frictional force changes when the sled is stopped. The friction becomes variable and takes on whatever magnitude is needed to offset the applied force, up to the maximum retarding force which the static friction can sustain.

Our simplified model is not designed to handle this case. One of the basic assumptions we made at the outset is that the sled is moving. If it is not moving, or once it stops moving, the equations we developed simply do not apply. One of the enhancements we will make in the more complex mathematical model is to add a second set of equations of motion to be used when the rig is stationary.

Just to complete this thought, I want to explain why this is not an issue at the starting gun. Yes, indeed, the coefficient and concept for static friction do apply when the rig is sitting back at the starting line and raring to go. But, we really don't care about the details of how the truck jolts the sled and gets it under way. We can just assume that it does. If the truck cannot budge the sled from the starting line, it surely will not win the competition.

Let me just mention time C identified in the acceleration's graph. If you look closely at the curve, you will see a slight change in the slope of the curve at that time. Point C occurs at time $t$ equal to 20 seconds. This is the moment when the moveable weight reaches its foremost point, and stops moving.
forward. Of course, this is only a mathematical oddity for the run we modeled, which came to an end before this event.

**Part IX -- An expression for the maximum distance travelled**

Simplified models like this one are not particularly useful for predicting exact results. They are much for useful for predicting how one change or another will affect the result. The result we are most interested in is the distance the rig can cover before it stops. Since we have a closed-form expression for the distance as a function of time, we can get a closed-form expression for the maximum distance reached.

If you look back at the first graph about, you will see that the moment in time when the rig reaches its maximum distance corresponds to the point where the slope of the distance-versus-time curve is horizontal. At times before this moment, the rig is moving forward, albeit ever more slowly. At times after this moment, the equations have it moving backwards. If we can identify the point on the distance-versus-time curve where it is horizontal, we will have identified the moment of maximum distance. If \( D = D(t) \) is the function representing the distance-versus-time curve, then its slope is the first derivative \( dD/dt \). The slope is horizontal when the derivative is equal to zero. In our analysis, we already encountered the derivative \( dD/dt \). It is the rig’s speed. So, we are looking for the moment in time when the speed becomes zero. Let's call this moment in time \( t_{\text{stop}} \). Equation (44) is the expression for the speed. At time \( t = t_{\text{stop}} \), the speed must be zero. We get:

\[
S(t) = \frac{1}{2}pt^2 + Qt
\]

At \( t = t_{\text{stop}} \),

\[
0 = \frac{1}{2}pt_{\text{stop}}^2 + Qt_{\text{stop}}
\]

\[
\Rightarrow 0 = \frac{1}{2}pt_{\text{stop}}^2 + Q
\]

\[
\Rightarrow t_{\text{stop}} = -\frac{2Q}{p} \quad (49)
\]

We can substitute this particular time directly into the distance function in Equation (48). We get:

\[
D(t_{\text{stop}}) = \frac{1}{6}pt_{\text{stop}}^3 + \frac{1}{2}Qt_{\text{stop}}^2
\]

\[
= \frac{1}{6}p \left( -\frac{2Q}{p} \right)^3 + \frac{1}{2}Q \left( -\frac{2Q}{p} \right)^2
\]

\[
= -\frac{8Q^3}{6p^2} + \frac{4Q^3}{2p^2}
\]

\[
= \frac{2Q^3}{3p^2} \quad (50)
\]

I will not bother to substitute the expressions for constants \( P \) and \( Q \) from Equation (43). To look at the sensitivities of maximum distance to certain values, it is easiest to transcribe the equations into an Excel spreadsheet. The plots in the following section we generated using Equation (50).
Part X -- Sensitivity analysis

Proposition #1 -- That increasing the weight of the truck at the rear will increase the run distance.

This may sound counter-intuitive, but consider this: If we add some weight to the truck, then it will have greater momentum when it reaches its maximum speed. That greater momentum will give it more "oomph" to drag the sled a few more feet near the end of the run.

The following plot shows what happens. The vertical axis is the maximum distance travelled by the rig during the run. It depends on the variable measured along the horizontal axis, which is the mass at the rear of the truck $M_{tr}$ which we have concentrated at the rear axle. In the base case, this mass is 2,040 pounds. I used Equation (50) to predict run distances for masses between 1,740 pounds and 2,340 pounds. The result of the base case is marked with a black dot.

Conclusion: It looks like intuition was right -- adding weight causes the rig to travel less far. The flaw in the proposition is that the truck's maximum speed is less. The extra momentum given by the extra mass is offset by the reduced momentum from lower speed.

Do note, though, that the distance is not very sensitive to the mass. Adding or removing one hundred pounds only changes the run distance by about one foot.

Adding weight has consequences which are not included in the simplified model. If the rear wheels slip, for example, added weight can increase the tractive force on the road and thereby increase the run distance. The simplified model does not permit wheel slipping. If the truck has a tendency to wheelie, added weight at the front may increase the tractive force. The simplified model does not allow for truck rotation, either.

![Run distance versus Rear-wheel weight](image-url)
Proposition #2 -- Increasing the size of the drive tires will increase the run distance.

This is obvious. Just look at farm tractors; they can pull heavy stuff and they have really big wheels.

The following plot shows what happens. As before, the vertical axis is the maximum distance travelled by the rig during the run, measured in feet. The variable measured along the horizontal axis is the so-called "deck height". In the simplified model, the deck height is set to the radius of the truck's rear wheels. The diameter of the wheels is twice the radius. In the base case, marked with a black dot, the wheels have a diameter of 31½ inches. The wheel diameters on which the plot is based range from about 26 inches to about 38 inches.

![Run distance versus Deck height](image)

Conclusion -- Boy, are farmers wrong. Reducing the wheel size increases run distance. Not just that, but the increase is substantial. The horizontal grid lines are spaced a distance 100 feet apart. Shaving off a half-inch of tread would be a useful thing to do before a truck pull.

So, why do farmers use big wheels? They have other objectives to think about, like driving over very rough fields and leaving clearance for growing crops. Big wheels decrease the maximum speed a vehicle can reach, but that is a secondary objective for farmers.
Proposition #3 -- Running a truck pull in second gear will increase the distance covered.

The following plot shows the effect of different transmission gear ratios. I have assumed that the lowest gear available in our stock pickup truck is first gear, which we used in the base case. This gear ratio, 2.84, defines the right-hand side of the horizontal axis. Higher gears have lower gear ratio. 1:1, which is near the left-hand side of the curve, is pretty standard for third gear.

The proposition is false; second gear is a flop. The rig would not even go 20 feet.

This proposition and the previous one are pointing us in the same direction. The objective is to maximize the horizontal force which the ground exerts on the rear wheels of the truck (in the free-body diagram). Things like smaller wheels and lower gears are ways to convert the torque available on the crankshaft in greater amounts of force on the ground.

But, the simplified mathematical model does not tell the whole story. There are other factors -- lots of other factors -- which all have their say. For example, the simplified model assumes the crankshaft torque is a constant from beginning to end during a run. This is a fine goal to aim for, but not so easy to reach. Racing the engine before the start of a run, and starting in second gear, may be ways to extract more torque from the crankshaft. Although that strategy is contrary to what works best for the dynamics of the rig, its impact on, and improvement to, the engine’s performance may mean it is the best strategy overall. To look into more complex effects like this one, a more complex model is needed.
Proposition #4 -- Boosting torque will increase the distance covered.

Truck pull enthusiasts spend a lot of time working to boost power, which tends to increase torque. The following plot shows what their efforts are worth. The horizontal axis in the plot is the crankshaft torque which, in the simplified model, is constant throughout the run. In all the cases shown, I used the same loss and efficiency parameters, and transmission and differential characteristics, as in the base case. Since enthusiasts would have little interest in reducing power, I only tested torques greater than those in the base case.

![Run distance versus Crankshaft torque](chart.png)

Roughly speaking, a 20 foot-pound increase will add 50 feet to the run distance. This corresponds to the second black dot on the plot, labeled "20 ft-lb boost".

Part XI -- Why truck pulls have become races

In this section, we are going to look at something a little different. We are going to change the crankshaft torque during a run, but only for a short period of time. In the base case, we assumed that the crankshaft torque was constant at 330 foot-pounds. What if, by some means unknown, we were able to increase the crankshaft torque to 400 foot-pounds, but only for one second. After the second was up, the torque would revert to 330 foot-pounds. The question is: when is the best time during a run to get that extra bit? The following figure compares two possibilities.
There was no magic in my picking 400 foot-pounds as the temporary boost. I just wanted to pick a torque large enough compared with the base case so that its effect, which only lasts one second, has a noticeable impact on the results. If the bump is too small, the changes are hard to see on the graphs which follow.

The only difference between the two cases is the time $t_{\text{boost}}$ at which the one-second bump commences. Do we want $t_{\text{boost}}$ to occur early in the run, or later?

Although the torque changes during the run, it does not change continuously. If we look at it the right way, the torque is always constant. But it is constant within three separate periods of time. We have a separate value of $\tau_{\text{crank}}^*^{\text{const}}$ within each time period.

$$\tau_{\text{crank}}^*^{\text{const}} = \begin{cases} 330 \text{ ft-lbs} & \text{when } t < t_{\text{boost}} \\ 400 \text{ ft-lbs} & \text{when } t_{\text{boost}} > t < t_{\text{boost}} + 1 \\ 330 \text{ ft-lbs} & \text{when } t > t_{\text{boost}} + 1 \end{cases} \quad (51)$$

Since the torque is constant within each separate time period, the analysis we did above to calculate the acceleration of the rig under constant torque remains valid, just so long as we apply it in each of the three time periods separately. The expression for the acceleration is set out in Equation (41). We defined constants $P$ and $Q$ to simplify the notation. The, during any period of time when the acceleration is constant, we can write $A(t) = Pt + Q$. The values of $P$ and $Q$ which apply during each time period will depend on the value of the torque during that time period. Looking back at the definition in Equation (43A), it is apparent that the constant $P$ does not depend on torque at all, and will remain constant during the entire run. Only $Q$ depends on torque, so we will use the symbols $Q_{330}$ and $Q_{400}$ for their values during the normal and boost periods, respectively.

We can then write the acceleration as the following piecewise function of time:

$$A(t) = \begin{cases} Pt + Q_{330} & \text{when } 0 < t < t_{\text{boost}} \\ Pt + Q_{400} & \text{when } t_{\text{boost}} < t < t_{\text{boost}} + 1 \\ Pt + Q_{330} & \text{when } t > t_{\text{boost}} + 1 \end{cases} \quad (52)$$

Within each time period, the acceleration can be integrated to give the speed. Rather than use definite integrals, I will integrate indefinitely and include constants to be determined by the initial conditions of each period.

$$S(t) = \begin{cases} \frac{1}{2}Pt^2 + Q_{330}t + S_1 & \text{when } 0 < t < t_{\text{boost}} \\ \frac{1}{2}Pt^2 + Q_{400}t + S_2 & \text{when } t_{\text{boost}} < t < t_{\text{boost}} + 1 \\ \frac{1}{2}Pt^2 + Q_{330}t + S_3 & \text{when } t > t_{\text{boost}} + 1 \end{cases} \quad (53)$$

A second integration gives the location of the rig at any time during each of the periods:

$$D(t) = \begin{cases} \frac{1}{6}Pt^3 + \frac{1}{2}Q_{330}t^2 + S_1t + D_1 & \text{when } 0 < t < t_{\text{boost}} \\ \frac{1}{6}Pt^3 + \frac{1}{2}Q_{400}t^2 + S_2t + D_2 & \text{when } t_{\text{boost}} < t < t_{\text{boost}} + 1 \\ \frac{1}{6}Pt^3 + \frac{1}{2}Q_{330}t^2 + S_3t + D_3 & \text{when } t > t_{\text{boost}} + 1 \end{cases} \quad (54)$$

The constants of integration are $S_i$ and $D_i$, for $i = 1,2,3$. Since the rig starts up from rest, $S(0) = 0$, and we can see that $S_1 = 0$. We cannot conclude that $S_2$ or $S_3$ are zero. The two expressions which contain
them do not apply during the period of time which includes \( t = 0 \). Similarly, since the rig is initially at rest, \( D_1 = 0 \). Philosophically, by looking at the conditions at the start of the first time period, we are able to compute the two constants \( D_1 \) and \( S_1 \) for that period.

We can do the same for the second time period. The initial conditions at the start of the second time period must be equal to the values of distance and speed which prevail at the end of the first time period. The equalities for the speed and distance at time \( t = t_{\text{boost}} \) are as follows:

\[
\begin{align*}
\text{Start of second period} &= \text{End of first period} \\
\text{For speed:} \quad &\frac{1}{2}pt_{\text{boost}}^2 + Q_{400}t_{\text{boost}} + S_2 = \frac{1}{2}pt_{\text{boost}}^2 + Q_{330}t_{\text{boost}} \\
\text{For distance:} \quad &\frac{1}{6}pt_{\text{boost}}^3 + \frac{1}{2}Q_{400}t_{\text{boost}}^2 + S_2t_{\text{boost}} + D_2 = \frac{1}{6}pt_{\text{boost}}^3 + \frac{1}{2}Q_{330}t_{\text{boost}}^2
\end{align*}
\]  

(55)

The "\( P \)" terms cancel out. From the first equation, we get:

\[ S_2 = -(Q_{400} - Q_{330})t_{\text{boost}} \quad (56A) \]

\( Q_{400} \) will be greater than \( Q_{330} \) so the constant \( S_2 \) will be algebraically negative. Substituting it into the second equation gives:

\[ D_2 = \frac{1}{2}(Q_{400} - Q_{330})t_{\text{boost}}^2 \quad (56B) \]

We can go through a similar dance at the start of the third time period, at which time \( t = t_{\text{boost}} + 1 \) the speed and distance given by the second- and third-period equations must be the same. We get, after canceling out the "\( P \)" terms:

\[
\begin{align*}
\text{Start of third period} &= \text{End of second period} \\
S: \quad &Q_{330}(t_{\text{boost}} + 1) + S_3 = Q_{400}(t_{\text{boost}} + 1) + S_2 \\
D: \quad &\frac{1}{2}Q_{330}(t_{\text{boost}} + 1)^2 + S_3(t_{\text{boost}} + 1) + D_3 = \frac{1}{2}Q_{400}(t_{\text{boost}} + 1)^2 + S_2(t_{\text{boost}} + 1) + D_2
\end{align*}
\]  

(57)

From the first equation, we get:

\[
S_3 = S_2 + (Q_{400} - Q_{330})(t_{\text{boost}} + 1)
\]

\[ = -(Q_{400} - Q_{330})t_{\text{boost}} + (Q_{400} - Q_{330})(t_{\text{boost}} + 1) = Q_{400} - Q_{330} \quad (58A) \]

Substituting this into the second equation gives:

\[
D_3 = \frac{1}{2}(Q_{400} - Q_{330})(t_{\text{boost}} + 1)^2 + (S_2 - S_3)(t_{\text{boost}} + 1) + D_2
\]

\[ = \frac{1}{2}(Q_{400} - Q_{330})(t_{\text{boost}} + 1)^2 - (Q_{400} - Q_{330})(t_{\text{boost}} + 1)^2 + \frac{1}{2}(Q_{400} - Q_{330})t_{\text{boost}}^2
\]

\[ = (Q_{400} - Q_{330})\left[-\frac{1}{2}(t_{\text{boost}} + 1)^2 + \frac{1}{2}t_{\text{boost}}^2\right] = -(Q_{400} - Q_{330})(t_{\text{boost}} + \frac{1}{2}) \quad (58B) \]

If you see certain patterns emerging here, you are right. The constants for each level can be expressed in terms of the constants of earlier levels and the forms of the expressions have much in common. If we were to divide the run up into more than three time periods, we would want to explore the pattern in more detail so as to avoid the period-by-period algebra.
Our primary interest is in the third time period. It is what follows after the boost that we are trying to work out. We can substitute the expressions for $S_3$ and $D_3$ into the last of Equations (53) and (54) to get self-contained expressions for the rig's speed and location during the third period. For times in the third period ($t > t_{\text{Boost}} + 1$), we get:

\[
S(t) = \frac{1}{2}pt^2 + Q_{330}t + (Q_{400} - Q_{330}) \quad (59A)
\]
\[
D(t) = \frac{1}{6}pt^3 + \frac{1}{2}Q_{330}t^2 + (Q_{400} - Q_{330})t - (Q_{400} - Q_{330})(t_{\text{Boost}} + \frac{1}{2}) \quad (59B)
\]

As a quick reality check, observe that the speed is the derivative of distance. This must always be the case. We can use the first expression, for speed, to compute the time which corresponds to the end of the run. At that time, the speed will be zero, so that:

\[
S(t_{\text{stop}}) = 0
\]
\[
\Rightarrow \quad \frac{1}{2}pt_{\text{stop}}^2 + Q_{330}t_{\text{stop}} + (Q_{400} - Q_{330}) = 0 \quad (60)
\]

This gives us the time; Equation (59B) evaluated at this time will give us the corresponding distance. Equation (60) is a quadratic equation in $t_{\text{stop}}$. Substituting its coefficients into the quadratic formula will give us the solution. Or, rather, the quadratic formula will give us two possible solutions, of which only one will correspond to the physical event of the rig's coming to a stop. The quadratic formula is:

\[
t_{\text{stop}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-Q_{330} \pm \sqrt{Q_{330}^2 - 2p(Q_{400} - Q_{330})}}{p} \quad (61)
\]

This does not depend on when the boost occurs! Whether it occur early during a run or later, the run will still end at the same time. I will not go through the computation of the factors, but the results are:

- $P = -0.0964$, with the same S.I. units as above,
- $Q_{330} = 0.922$, with the same S.I. units as above,
- $Q_{400} = 1.172$, with the same S.I. units as above, with the result that
- $t_{\text{stop}} = 19.4$ seconds.

The run takes a little bit longer than it did in the base case, approximately half a second longer. The rig is given a little bit more energy during the run so it travels a little bit further. Travelling further takes a little bit longer.

Substituting the stop time $t_{\text{stop}}$ into Equation (59B) will gives lots of cubes of square roots and stuff like that. A visual display is more informative. The following graph shows the stopping times which correspond to various possible commencement times for the boost.
The horizontal axis shows the time at which the boost starts. I have blocked off parts of the graph where the boost starts later than about 18.4 seconds into the run, because that means the run is over before the one-second boost is finished.

The vertical axis shows the distance covered by the rig during the run. The sooner the boost occurs during a run, the more benefit it confers on the rig.

This is not an intuitively obvious result. An alternative argument goes like this. Since work is the magnitude of a force multiplied by the distance the forced object moves, power is the magnitude of the force multiplied by the speed at which the object moves. More power is added to the object is a given amount of energy is added when the object is moving fastest. Since the rig is moving fastest at about the mid-point of the run, this argument goes, the boost should be added then. And, indeed, this argument is correct, assuming that all else is equal.

But, all else is not equal during a truck pull. Something else -- a very important something else -- changes with time. This is the location of the moveable weight. It moves inexorably forward and makes the sled harder to drag as time passes.

When all the factors are taken into account, it seems to make more sense to apply the boost early in the run while the sled is still easy to drag. I have heard some truck pull enthusiasts complain that truck pulls have turned into truck races over the past couple of decades. Whether they like races or not, the underlying physics favour a winner who gets the sled moving as fast as possible while the going is easy.

Let me finish this section by showing some of the kinematic curves for various boost times. This first curve shows the acceleration of the rig through time as the run takes place. I have shown four curves, one when the boost starts at time $t_{boost} = 0$, a second when the one-second boost starts five seconds into the run ($t_{boost} = 5$), a third when the boost starts at $t_{boost} = 10$ and the last when the boost starts 15 seconds into the run.
The acceleration develops in the same way over time in all the cases except during the one second when the boost occurs. The boost, representing the addition of energy to the rig, accelerates the rig. That is not to say that the rig must have positive acceleration when the boost occurs. The addition of energy increases an already positive acceleration or decreases an existing deceleration. The rig has a net positive acceleration at the start of the run. A positive acceleration is the only way the rig can start moving forward from a standing start. But, the acceleration decreases steadily as the moveable weight makes the sled harder and harder to pull. For a substantial part of the run, the rig is actually decelerating towards its stop.

The next graph shows the speed of the rig during the same four runs.
As we already found, the speed returns to zero at the same time without regard to when the boost took place. The speed returning to zero is the event which marks the end of the run. The rig will not move any further.

This graph makes it a little clearer why it is beneficial to have the boost early in the run. Once the boost has added some speed, the extra speed lasts for the remainder of the run. The sooner the speed is boosted, the longer the extra speed lasts.

The next graph is the corresponding distance trajectories. The location of the rig is traced through the run for the same four cases whose acceleration and speed were shown above.

![Distance trajectories for various boost times](image)

All four distance curves peak at the same moment in time and are horizontal there. All the runs end at the same time. But, the sooner the boost was applied, the further the rig pulled out ahead of the other cases. Let me tie this result back into the base case. In the base case, the rig traveled about 184 feet. We can get that up to about 200 feet by applying a one-second boost at the start of the run. The boost was from 330 to 400 foot-pounds, or an increase of 70 foot-pounds, at the crankshaft. More grist for the mill of those enthusiasts for whom an insane amount of power is not quite enough.

That is enough work for a simple model. It is time to move ahead and look at a more complex model for dynamics.

Jim Hawley
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An e-mail setting out errors or omissions would be appreciated.