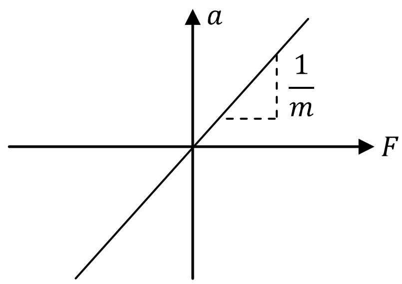
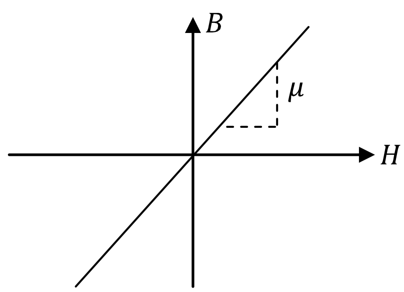
Modeling a cylindrical permanent magnet with a surface charge of magnetic monopoles

Let us begin somewhere else – with Newton's Laws. I have illustrated Newton's Third Law for a rigid body in the following graph.



The horizontal axis is the force applied to the rigid body. The vertical axis is the body's response. It accelerates. The magnitude of the acceleration is proportional to the magnitude of the applied force, and the constant of proportionality is equal to $1/m$, where m is the body's mass.

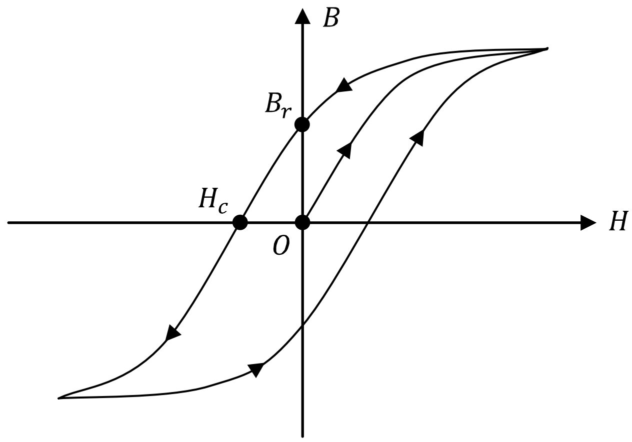
Something similar occurs when a "magnetic" force is applied to any material. The material may respond by generating its own magnetic field, which will be in addition to the applied magnetic field. The response curve in this case is called a "magnetization curve" and looks something like this.



There are several things to note:

1. The "magnetic" force is also called the "magnetomotive" force. It is commonly represented by the symbol H. In the SI system, H is measured in $\text{Ampere-turns/meter}$. Amperes and turns are quantities one uses when thinking about solenoids. This is, in fact, the historical origin of this usage. Early investigators used solenoids to apply the H-field and, when it came time to standardize units of measurement, their practice was adopted.
2. This relationship applies to "any material" as distinct from a "body". A magnetization curve applies to a specific kind of substance, and does not depend on the size or shape of the sample. On the other hand, the sample cannot be microscopic. The B vs H relationship is a statistical average of the response of all the electrons in the sample. While it arises because of the response of the individual bound and free electrons in the sample, it is not a useful relationship for examining the behavior of individual electrons.
3. It is assumed that the material is capable of responding to external magnetic forces. Non-magnetic materials, like wood, simply do not respond.
4. The response of the material is commonly represented by the symbol B. In the SI system, B is measured in Tesla, where 1 Tesla $= 1\ \text{Weber/meter}^2$. That the denominator is an area tells us something about B – it is the density of something per unit area. The something in this case are the lines of magnetic flux. In a strong magnetic field, there are lots of lines of magnetic flux per unit area. For this reason, B is also referred to as the "magnetic flux density".

Let us look at a typical magnetization curve, as shown in the following graph. The arrows show the state of the material as the H-field is changed.



At the outset, before any external magnetic field is applied, the material is in the state labeled O in the figure. When an external magnetic field is applied, a magnetic flux density is induced in the material. As the strength of the external field is increased, the induced magnetic flux density increases as well. The state of the material moves from the origin O towards the upper right. Often, there is a range where the induced magnetic flux density increases almost linearly with the applied field strength. If this is so, the constant of proportionality μ in the relationship $B = \mu H$ is called the "permeability" of the material. If the relationship is not linear, the slope of the B vs H curve at the origin O is called the "initial" permeability and is often represented by the symbol μ_i.

For all materials, the increase in B per unit increase in H will eventually begin to decrease. The slope of the curve falls off as H is increased without bound. By considering the material at the microscopic scale, it is easy to see why this must be the case. The induced flux density B is a macroscopic quantity which arises from the velocities and spins of huge numbers of electrons in the material. Some of the electrons may be "free", by which we mean that they are so loosely bound to their mother atoms that they are able to move about in response to the applied magnetic field H. The force exerted by a magnetic field on a charged particle (like an electron) acts in a direction which is perpendicular to its velocity. Such a force will cause the electron to travel in a circle whose central axis is aligned which the local direction of the applied field. However, an electron traveling in a circle is like a current flowing through a circular loop – both generate a magnetic field whose direction is aligned with the central axis of the circle. Each orbiting electron will generate its own little magnetic field. As the strength of the applied magnetic field is increased, an increasing number of electrons will become free from their mother atoms and be able to orbit. The orbits of the free electrons will become tighter and tighter and the magnetic fields each of them generates will increase. Furthermore, their orbits will become increasingly aligned with the external magnetic field. All of these factors contribute to the increase in the magnetic flux density.

A second factor is at work, too. The atomic spins of the electrons, both free and bound, are magnetic quantities. They, too, can and do respond to the external magnetic field. As H is increased, the spin axes of more electrons will become aligned with the local direction.

There comes a point when the electrons in the material have responded as much as they are able. Further increases in the strength of the applied magnetic field do not increase the strength of the induced magnetic flux density any further. At this point, the material is said to be fully "saturated". This state corresponds to the vertex at the upper-right of the B vs H curve.

From this saturation point, let us now begin to reduce the strength of the applied magnetic field. All magnetic materials exhibit some degree of "hysteresis". Hysteresis means that the material will not retrace its path back to the origin O. As the external field is reduced, the free electrons will retain some characteristics of the motion which they adopted when the external field was first applied and then increased. This will cause the induced flux density B to be greater at each particular value of H than it was on "the way up". Geometrically, this places the B vs H curve on "the way down" above the curve on the way up.

Even when the external field strength is reduced to zero, corresponding to the vertical axis in the graph above, the material will retain some magnetic flux density. This is the state labeled B_r in the figure above, where the subscript r stands for residual.

In fact, in order to reduce the magnetic flux density to zero, it is necessary to reverse the direction of the external field. The value of the H-field needed to reduce the B-field to zero is called the "coercive" force and is represented by the symbol H_c. When the magnetic flux density is forced to zero, the material is in the state labeled H_c in the figure above. H_c is sometimes also called the "critical magnetizing force".

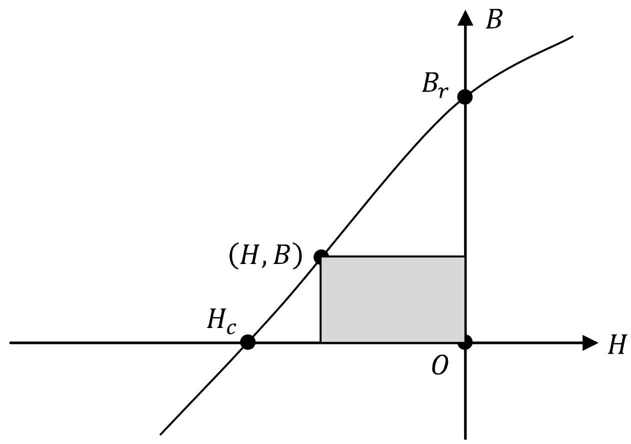
Of course, there will be some value of the externally-applied reverse magnetizing force at which the magnetic flux density in the reverse direction cannot be increased further. This state corresponds to the vertex at the lower-left of the B vs H curve. If the material is isotropic, that is, if its physical properties are the same in all directions, then the vertex at the lower-left will be symmetric through the origin O with the vertex at the upper-right.

Applying an external magnetizing force is a popular way to make a piece of steel into a permanent magnet. A permanent magnet "operates" in the upper-left quadrant of its B vs H curve. The part of the magnetization curve from point B_r to point H_c is called the "operating region" of a permanent magnet. When one holds the magnet in the hand, away from other metals, it has a magnetic flux density of B_r. The maximum magnetizing force it can exert when placed on the surface of a magnetic material is H_c.

In recent years, "rare-earth" magnets have become very popular. They are called "rare-earth" because they are manufactured from alloys of the Lanthanide group of elements in the periodic table, which were historically called the rare-earths. The two most common elements in the Lanthanide group are neodymium (Nd) and samarium (Sm). They are usually alloyed with iron, boron or cobalt to manage other characteristics such as brittleness, temperature resistance or corrosion resistance.

Rare-earth magnets have replaced ceramic (ferrite) magnets and Alnico (aluminum + nickel + cobalt) magnets in many consumer applications because they are small and handy and, most importantly, "strong". They are ideal for refrigerator, filing cabinet and cubicle use. They are often sold as little cylinders, which makes them easier to remove from refrigerator, filing cabinet or cubicle frame.

Two parameters are usually specified to describe the "strength" of a permanent magnet. They are B_r, which we have already seen, and BH_{max}, which we have not. BH_{max} is the "Maximum Energy Product". It is a measure of the material from which the magnet is made and is not related to the size or shape of the magnet. For this reason, the Maximum Energy Product is often called the "grade" of a permanent magnet. The Maximum Energy Product has a geometrical interpretation. It is the maximum of the products of B and H along all the points on the B vs H curve between points B_r and H_c. In other words, it is the maximum area of all rectangles of the type shown shaded in the following figure.



A company called Forcefield sells a small NdFeB (neodymium + iron + boron) rod magnet, ¼-inch in diameter and ¼-inch in length. Its specification sheet reports $B_r = 1.27$ Tesla and $BH_{max} = 40$ MGOe. The unit MGOe is a mega Gauss-Oersted, or one million Gauss-Oersteds. This is not a metric or SI unit, but is used too often to quibble. A magnetic material with this strength would be referred to as a "grade 40" material. Since 1 Tesla $= 10{,}000$ Gauss, the Maximum Energy product of this material would be expressed in SI units as $4{,}000$ TOe.

Magnetization

The microscopic interactions I described above give rise to the "magnetization" of the material. The magnetization is the counterpart of the "polarization" talked about for electric fields. The magnetization arises from many small circulations of electron "current", all circulating in the same sense about the direction of the externally applied magnetic field. In many cases, a greater proportion of the material's response comes from alignment of the electrons' spins rather than their motion in orbits aligned with the external field.

Let us consider the case when a body is immersed in a uniform applied magnetic field. Let us consider two small elements of volume somewhere inside the body and which are adjacent. We can think of the electrons' response as being a circulation of current around the outer surface of each volume element, in a direction which is perpendicular to the local direction of the applied magnetic field. Since both volume elements consist of the same material, the response of the electrons will be the same inside them both and the circulation of current will be the same around the outer surfaces of them both. Since the two volume elements are adjacent, they share a common boundary. On the common boundary, the current around one volume element will be offset by an equal and opposite current around the other. The currents will cancel each other out on the boundary between any two neighbouring volume elements.

Therefore, at any point in the interior of the body, there is no net current in any plane perpendicular to the applied magnetic field. Only on the surface of the material, where there is no cancellation of current flowing on the "outside" boundaries of the volume elements on the surface, will there be any net flow of current. The net current in the case of a macroscopic body will flow on the surface.

If the body is a cylinder, and if the magnetization is aligned with the longitudinal axis of the cylinder, then the net current will flow only on the curved surface of the cylinder. There will be no net current flowing on the end faces, which are assumed to be flat and to be perpendicular to the direction of the applied field / magnetization. Furthermore, the net current flowing on the curved surface will, in fact, be in circles which are perpendicular to the longitudinal axis, since this is the direction which is perpendicular to the applied field / magnetization. Geometrically, the net current will be the same as it is in a cylindrical solenoid. This is, of course, one approach to making a mathematical model of a cylindrical magnet.

Let us return to the body immersed in an applied magnetic field, the H-field. The total magnetic flux density B_{total} at any point in the body is the sum of two things: (i) the applied magnetic field, which exists whether the body is present or not, plus (ii) the additional magnetic field generated by the material in response to the external field. This additional magnetic field is called the "induced" field. We can write the magnetic flux density at any point as follows:

$$\vec{B}_{total} = \vec{B}_{applied} + \vec{B}_{induced} \quad (1)$$

We can find $\vec{B}_{applied}$ by removing the body from the scene. When the point in question is in free space, or in air, the flux density of the H-field is given by:

$$\vec{B}_{applied} = \mu_0 \vec{H} \quad (2)$$

where μ_0 is the permeability of free space, being $\mu_0 = 4\pi \times 10^{-7}$ Ampere-turn-meters. The units I have given for μ_0 are consistent with the units of measurement described above for H and B. But, these units can be expressed in other ways using other derived units. For example, an Ampere-turn-meter is the same as one Volt-second/Ampere- meter, one Henry/meter, one Newton/Ampere^2, one Tesla-meter/Ampere or one Weber/Ampere-meter.

Now, let us bring the body back. It is convenient to define vector $\vec{M}$ as the magnetic moment per unit volume at any point in the material. The magnetic flux density $\vec{B}_{induced}$ induced by the applied field is related to the magnetization $\vec{M}$ in a straightforward way:

$$\vec{B}_{induced} = \mu_0 \vec{M} \quad (3)$$

Therefore, the total magnetic field $\vec{B}_{total}$ can be written as:

$$\vec{B}_{total} = \mu_0 \vec{H} + \mu_0 \vec{M} \quad (4)$$

For materials where H_c and B_r are relatively small (that is, do not make good permanent magnets), the magnetization $\vec{M}$ is proportional to the $\vec{H}$-field, and can be expressed as:

$$\vec{M} = \chi \vec{H} \quad (5)$$

where χ is called the magnetic susceptibility of the material. Then, substituting successively from above, we get:

$$\vec{B}_{total} = \mu_0 (1 + \chi) \vec{H} \quad (6)$$

The relative permeability μ_r of the material is defined to be equal to $1 + \chi$, so that:

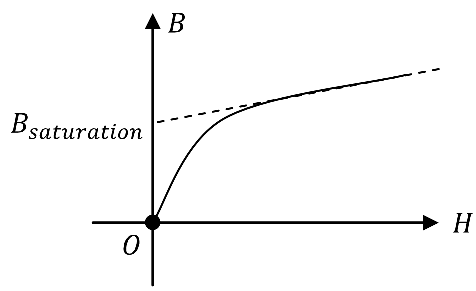
$$\vec{B}_{total} = \mu_0 \mu_r \vec{H} \quad (7)$$

For these materials, one can calculate $\vec{B}_{total}$ merely by multiplying $\vec{H}$ by $\mu_0\mu_r$. It is important to understand that the material "amplifies" the strength of the applied magnetic field, but does not change its direction. At any point in the material, the electrons' response is aligned with the external field at that point, and the additional contribution to the magnetic field from the electrons' response is also aligned with the external field at that point, with the result that the total magnetic field has a greater magnitude but the same direction as the applied field.

A perennial source of difficulty is that μ_r is not a constant. It depends on the strength of the H-field. For relatively small external fields, the relative permeability may be constant or nearly so. As the strength of the external field increases, the relative permeability will decrease. Eventually, after the material becomes fully saturated, there is no further increase in $\vec{B}_{induced}$, and $\vec{B}_{total}$ increases as $\mu_0\vec{H}$ only. In fact, for many materials, there is a so-called "saturation density" $\vec{B}_{saturation}$ (which is not necessarily the same as the fully-saturated vertex described above) for which one can write:

$$\vec{B}_{total} = \vec{B}_{saturation} + \mu_0\vec{H}, \text{for large applied fields} \quad (8)$$

Equation (8) applies only when the applied field is relatively strong. It is a linear approximation of the B vs H curve which is illustrated in the following graph.



A perennial source of confusion when a B is given, is to sort out whether the quantity includes the contribution of the external field (and, so, is really $\vec{B}_{total}$) or does not include the applied field (and, so, is really $\vec{B}_{induced}$). Similar confusion can arise when a permeability μ_r is given – is it χ or $\chi + 1$?

Typical parameters of a permanent magnet

We saw above that, for a cylindrical rod of material, with the external magnetic field aligned along the longitudinal axis, the magnetization is identical to that produced by a solenoid. If the turns of the solenoid are ideally thin, then the current flowing through the succession of turns of wire is the same as the sheet of current circulating around the surface of the cylindrical magnet.

In the paper titled *The magnetic field in and around a finite cylindrical air-core solenoid*, we calculated that the magnetic flux density at the geometric center of a solenoid is approximately equal to:

$$\vec{B}\Big|_{center} \cong \frac{\mu I N_{turns} N_{layers}}{H_{solenoid}}\hat{z} \quad (9)$$

In this expression, the $\hat{z}$-axis is the longitudinal axis of the solenoid, N_{turns} and N_{layers} are the number of turns and layers, respectively, carrying current I and $H_{solenoid}$ is the length of the solenoid. The permeability μ is the permeability of the material which occupies the geometric center of the solenoid.

This expression applies when the solenoid is long compared to its radius. The approximation gets better as the coil becomes longer and thinner, and becomes exact for an infinitely-long solenoid. If we let J be the current per unit length of the solenoid, then:

$$J = \frac{I N_{turns} N_{layers}}{H_{solenoid}} \quad (10)$$

and the flux density can be written as:

$$B = \mu J \quad (11)$$

The question now arises: how do we find out the "magnetization" of a permanent cylindrical magnet so that we can calculate the equivalent current density?

The following table sets out some parameters of common permanent magnet material. The material and grade are self-explanatory. B_r, H_c and BH_{max} are the parameters from the B vs H curve described above. In the table, B_r and H_c are given in cgs units, as Gauss and Oersteds, respectively. T_{max} is one manufacturer's recommended maximum practical operating temperature, in degrees centigrade.

Material	**Grade**	$\boldsymbol{B_r}$	$\boldsymbol{H_c}$	$\boldsymbol{H_{ci}}$	$\boldsymbol{BH_{max}}$	$\boldsymbol{T_{max}}$
NdFeB	39H	12,800	12,300	21,000	40	150
SmCo	26	10,500	9,200	10,000	26	300
NdFeB	B10N	6,800	5,780	10,300	10	150
Alnico	5	12,500	640	640	5.5	540
Ceramic	8	3,900	3,200	3,250	3.5	300
Flexible	1	1,600	1,370	1,380	0.6	100

H_{ci} arises in the following way. We have already seen that the total magnetic field B at any point in a magnetic material is equal to the sum of the applied field H and the induced field B_i produced by the intrinsic ability of the material to amplify magnetic flux. Then, $B = H + B_i$, or $B_i = B - H$. Under normal operating conditions, no external magnetizing field is present, and the permanent magnet operates in the second quadrant, where H has a negative value. A negative value is inconvenient for some purposes. Accordingly, practitioners prefer to use a positive value of H so, for them, $B_i = B + H$. One can plot a B_i vs H curve using these values of B_i. If this is done, then H_{ci} is the ordinate where the curve crosses the H-axis. (Note that it is possible to derive the B_i vs H curve from the B vs H curve, and vice versa.) High values of H_{ci} are an indicator of inherent stability of the magnet material.

The residual flux density at the center of a solenoid used to model a cylindrical permanent magnet, if made from one of these materials, should be equal to the residual flux density B_r in the table above. For a solenoid which produces the same magnetic field as a grade 39H rare-earth magnet, then:

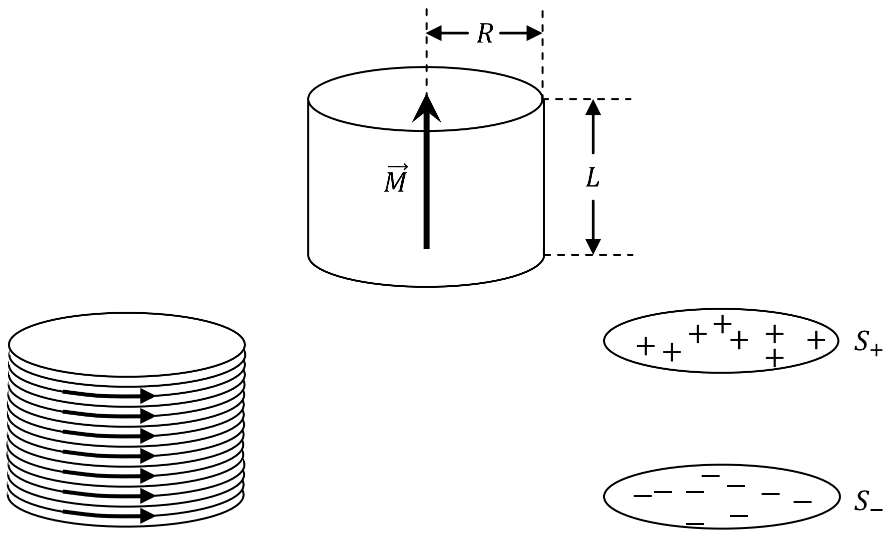
$$\begin{aligned} J &= \frac{B_r}{\mu_0} \\ &= \frac{12{,}800 \text{ Gauss}}{\mu_0} = \frac{1.28 \text{ Tesla}}{\mu_0} \\ &= \frac{1.28}{4\pi \times 10^{-7}} \text{ A/m} \\ &= 1.02 \times 10^{6} \text{ A/m} \end{aligned}$$

That is a lot of current.

Source models for a cylindrical permanent magnet

A "source" model is a mathematical model from which one can make calculations. Since the magnet is an electromagnetic object, one would expect that its relevant characteristics could be related to more fundamental electromagnetic concepts like charge and current. The goal of this paper is to find out what these relationships are. It turns out that there are two mathematical models one can use.

Let us assume that the magnet has a uniform axial magnetization $\vec{M}$. The length and radius of the magnet are given by L and R, respectively. The following figure shows the physical magnet and its two equivalent source models.



Model with current sheet $\vec{J}$ Model with surface monopoles σ^*

On the left is a thin-wall solenoid with length L and radius R, and having current density $\vec{J}$. It produces exactly the same magnetic field as the permanent magnet if and only if:

$$\text{Current model:} \quad \vec{J} = \mu_0 \vec{M} \quad (12)$$

On the right are two ideal circular plates with radius R, separated by distance L, and having magnetic monopole "charge densities" of $+\sigma^*$ and $-\sigma^*$ on their surfaces. It produces exactly the same magnetic field as the permanent magnet if and only if:

$$\text{Monopole model:} \quad \sigma^* = \vec{M} \cdot \hat{n} \quad (13)$$

where $\hat{n}$ is the unit vector normal to the plates.

The representation on the left can be approximated using physical components, like a solenoid. Because there are no such things as magnetic monopoles, the one on the right cannot. Even so, it produces the same magnetic field as the permanent magnet. While both representations produce the same result, they are processed quite differently from a mathematical point-of-view. It happens, therefore, that a given physical configuration might be solved more easily using one model rather than the other.

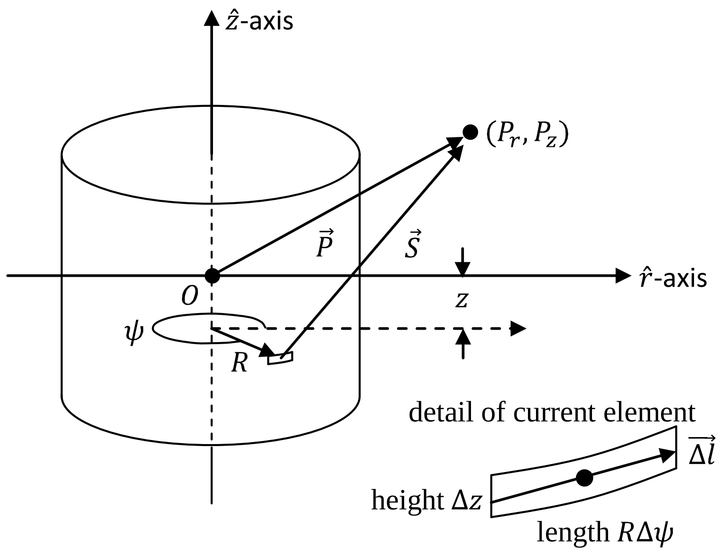
The current sheet model

Consider the representation on the left in the figure above, where the permanent magnet is represented by a current sheet. The current sheet is ideally thin and coincident with the curved surface of the magnet.

Messrs. Biot and Savart were the first to describe rigorously the magnetic flux density $\overrightarrow{\Delta B}$ generated by a very small straight length of current-carrying wire at some point removed by vector $\vec{S}$ from the center of the current element. They found that:

$$\overrightarrow{\Delta B} = \frac{\mu_0 I \overrightarrow{\Delta l} \times \vec{S}}{4\pi S^3} \quad (14)$$

where the multiplication of vectors in the numerator is the vector cross-product. The following figure shows how we will divide the current sheet around the cylinder into small length elements to which we can apply Biot-Savart's Law.



The origin O of the co-ordinate frame of reference is located at the geometric center of the cylinder. The $\hat{z}$-axis is coincident with the longitudinal axis of the cylinder. Without any loss of generality, we can select a radial axis $\hat{r}$ so that the $\hat{r}$-$\hat{z}$ plane includes the point of interest $\vec{P}$, at which we want to calculate the magnetic field. The location of point $\vec{P}$ is specified by its co-ordinates (P_r, P_z) along the two axes. We could call the third axis, which will be perpendicular to other two, the $\hat{y}$-axis. I have not shown the $\hat{y}$-axis in the figure. It will point from the origin directly into the page.

We are going to look at very small elements of area on the surface of the cylinder. Since all such elements lie on the surface, they are all a distance R from the $\hat{z}$-axis. A typical small element is identified by the distance z above the $\hat{r}$-$\hat{y}$ plane and by the angle ψ by which is rotated around the $\hat{z}$-axis in the right-hand direction. The particular element of area shown on the surface in the figure above is located somewhere along the negative $\hat{z}$-axis, so its z-co-ordinate is algebraically negative. Its angle ψ is about $300°$.

The small elements of area we will consider are small rectangular arc sections of the cylinder. They have a height of Δz and an arc length of $R\Delta\psi$, where angle $\Delta\psi$ is the angle subtended by the element when it is viewed from the $\hat{z}$-axis. As the dimensions of the element of area are made smaller and smaller, its curvature in the circumferential direction becomes less and less significant. Eventually, when it has small enough dimensions, we can consider the element to be a flat rectangle, with height Δz and length $R\Delta\psi$. The current flows along the length, parallel to the $\Delta\psi$-direction. In essence, the element of area is a small straight piece of wire, with length $\Delta l = R\Delta\psi$. Suppose we also assign a direction to the rectangle, being the direction in which the current flows along it. Then, $\overrightarrow{\Delta l}$ is a vector with length Δl and a direction which is tangent to the surface of the cylinder (at the center of the rectangle) and oriented circumferentially.

The area of the small element of area, or rectangle, is given by:

$$\begin{aligned}\Delta\text{Area} &= (\text{vertical height})(\text{horizontal width}) \\ &= (\text{vertical height})(\text{circumference of cylinder})(\text{fraction of circle}) \\ &= \Delta z(2\pi R)\left(\frac{\Delta\psi}{2\pi}\right) \\ &= R\Delta z\Delta\psi \qquad (15)\end{aligned}$$

In the co-ordinate frame being used, the components of the tangential vector $\overrightarrow{\Delta l}$ can be written as:

$$\overrightarrow{\Delta l} = -\Delta l \sin\psi\,\hat{r} + \Delta l \cos\psi\,\hat{y} \qquad (16)$$

We have now defined almost enough symbols for us to apply Biot-Savart's Law in Equation (14). Let us define J as the "current density" flowing around the cylinder. This is the amount of current (in Amperes) flowing per unit length (in meters) along the axis of the cylinder. The amount of current flowing through the element of area is then given by:

$$\begin{aligned}\text{current} &= (\text{current density})(\text{vertical height}) \\ &= J\Delta z \qquad (17)\end{aligned}$$

One thing remains, and that is to find the vector $\vec{S}$, which points from the center of the element of area to the point of interest $\vec{P}$. We can derive $\vec{S}$ using vector subtraction. In the co-ordinate frame being used, the components of point $\vec{P}$ can be written as:

$$\vec{P} = P_r\hat{r} + P_z\hat{z} \qquad (18)$$

Then, let vector $\vec{Q}$ point from the origin O to the center of the current element. The components of vector $\vec{Q}$ can be written as:

$$\vec{Q} = R\cos\psi\,\hat{r} + R\sin\psi\,\hat{y} + z\hat{z} \qquad (19)$$

Using vector subtraction, the vector $\vec{S}$ from the center of the current element to the point of interest is:

$$\vec{S} = \vec{P} - \vec{Q} = (P_r - R\cos\psi)\hat{r} - R\sin\psi\,\hat{y} + (P_z - z)\hat{z} \qquad (20)$$

The square of the length of $\vec{S}$ is given by:

$$\left|\vec{S}\right|^2 = (P_r - R\cos\psi)^2 + R^2\sin^2\psi + (P_z - z)^2 \qquad (21)$$

To apply Biot-Savart's Law, we need to calculate the vector cross-product $\overrightarrow{\Delta l} \times \vec{S}$. It can be found by evaluating the following determinant:

$$
\begin{aligned}
\overrightarrow{\Delta l} \times \vec{S} &= \begin{vmatrix} \hat{r} & \hat{y} & \hat{z} \\ -\Delta l \sin\psi & \Delta l \cos\psi & 0 \\ (P_r - R\cos\psi) & -R\sin\psi & (P_z - z) \end{vmatrix} \\
&= \Delta l (P_z - z)\cos\psi\,\hat{r} + \Delta l (P_z - z)\sin\psi\,\hat{y} + \Delta l[R\sin^2\psi - (P_r - R\cos\psi)\cos\psi]\hat{z} \\
&= \Delta l (P_z - z)\cos\psi\,\hat{r} + \Delta l (P_z - z)\sin\psi\,\hat{y} + \Delta l (R - P_r\cos\psi)\hat{z} \qquad (22)
\end{aligned}
$$

Let us use the symbol $\overrightarrow{\Delta B}$ for the total magnetic flux density generated by this element of area. For this element, Equation (14) becomes:

$$
\begin{aligned}
\overrightarrow{\Delta B} &= \frac{\mu_0 \text{current}}{4\pi|\vec{S}|^3} \overrightarrow{\Delta l} \times \vec{S} \\
&= \frac{\mu_0 (J\Delta z)}{4\pi} \frac{(R\Delta\psi)[(P_z - z)\cos\psi\,\hat{r} + (P_z - z)\sin\psi\,\hat{y} + (R - P_r\cos\psi)\hat{z}]}{[(P_r - R\cos\psi)^2 + R^2\sin^2\psi + (P_z - z)^2]^{3/2}} \qquad (23)
\end{aligned}
$$

Each small element of area into which we divide the surface of the cylinder generates a small flux density at point $\vec{P}$, in the amount of $\overrightarrow{\Delta B}$ in Equation (23). To find the total flux density $\vec{B}$ at point $\vec{P}$, one adds up the contributions made by all the small elements of area on the surface of the cylinder. The sum can be written as follows:

$$
\vec{B} = \sum_{all\ \Delta z} \sum_{all\ \Delta\psi} \overrightarrow{\Delta B} \qquad (24)
$$

In the limit as the dimensions of the elements of area become small enough to be treated as differentials dz and $Rd\psi$, instead of differences Δz and $R\Delta\psi$, the summation can be written as the following integral:

$$
\vec{B} = \frac{\mu_0 J R}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + (P_z - z)\sin\psi\,\hat{y} + (R - P_r\cos\psi)\hat{z}}{[(P_r - R\cos\psi)^2 + R^2\sin^2\psi + (P_z - z)^2]^{3/2}} d\psi dz \qquad (25)
$$

A trigonometric identity can be used to simplify the denominator, leaving:

$$
\vec{B} = \frac{\mu_0 J R}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + (P_z - z)\sin\psi\,\hat{y} + (R - P_r\cos\psi)\hat{z}}{[P_r^2 + R^2 + (P_z - z)^2 - 2RP_r\cos\psi]^{3/2}} d\psi dz \qquad (26)
$$

Now, it turns out that the component of $\vec{B}$ in the $\hat{y}$-direction vanishes. Consider the traverse from $\psi = 0$ to $\psi = 2\pi$ for those elements of area which have some constant displacement z along the $\hat{z}$-axis. In other words, consider the integral over ψ when z is held fixed. The denominator of the integrand is symmetric in ψ, because $\cos(-\psi) = \cos\psi$. But, the $\hat{y}$-term in the numerator is asymmetric, because $\sin(-\psi) = -\sin\psi$. The contributions to $(P_z - \mathrm{z})\sin\psi$ made by two diametrically-opposed current elements will cancel each other out. This reduces the expression for $\vec{B}$ to:

$$\vec{B}_{current} = \frac{\mu_0 JR}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + (R - P_r\cos\psi)\hat{z}}{\left[P_r^2 + R^2 + (P_z - z)^2 - 2RP_r\cos\psi\right]^{3/2}} d\psi dz \qquad (27)$$

I have identified this magnetic field with a subscript $current$ to note that it is the result from a mathematical model of a current sheet. This will distinguish it from the result found using the magnetic monopoles, or charges, for which we will use the subscript $charge$.

The magnetic monopole model

Now, let us consider the other mathematical model for a cylindrical permanent magnet, which was illustrated on the right-hand side of a figure some pages past. In this model, the magnet is represented by two circular and parallel plates charged with magnetic monopoles. The two plates are ideally thin and, physically, are coincident with the end-faces of the cylindrical magnet.

Messr. Coulomb was the first to describe rigorously the electric field $\vec{E}$ generated by a single point charge having charge q_e. At a point of interest located by vector $\vec{S}$ from the charge, the electric field is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_e}{S^3} \hat{S} \qquad (28)$$

where ϵ_0 is the permittivity of free space. Messr. Coulomb's electric field satisfies one of Maxwell's Equations, namely:

$$\vec{\nabla} \cdot \vec{E} = q \qquad (29)$$

Messr. Dirac was the first to extend this treatment of electric charges to magnetic monopoles. Although magnetic monopoles did not exist in Dirac's day, and have still not been found experimentally, his insight was extremely useful. He imagined a magnetic monopole with magnetic charge q_m and proposed that the magnetic field generated by this single point monopole at a point of interest located by vector $\vec{S}$ away, would be given by:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q_m}{S^3} \hat{S} \qquad (30)$$

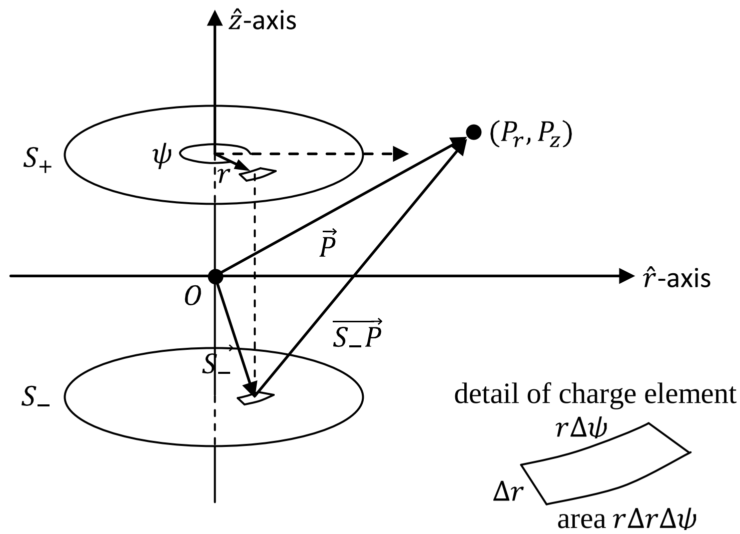
where μ_0 is the permeability of free space. (Curious readers may wonder why the constant μ_0 is not in the denominator, where we could have left it, so that Equations (28) and (30) have exactly the same logical form. All that is really important is that the units of q_e, q_m, ϵ_0 and μ_0 all be consistent for their positions in these equations. Well, it turns out that ϵ_0 and μ_0 are related to the square of the speed of light by $\epsilon_0\mu_0 = c^2$, so we can simplify the units of q_m if we put its constant in a location where it "opposes" ϵ_0.) Now, Messr. Dirac's magnetic field satisfies the equation:

$$\vec{\nabla} \cdot \vec{B} = q_m \qquad (31)$$

This is not the traditional Maxwell Equation, which is $\vec{\nabla} \cdot \vec{B} = 0$, based on the presumption that magnetic monopoles do not exist. To be precise, both Equations (29) and (31) should be understood to apply everywhere except "within" the point charge or magnetic monopole itself.

Whether or not magnetic monopoles exist, the magnetic field $\vec{B}$ in Equation (30) can be manipulated in exactly the same way as the electric field$\vec{E}$ in Equation (28).

The following figure shows the monopole source model which represents the cylindrical magnet. The co-ordinate frame of reference is the same as that used above, with the origin O now located at the geometric center of the two circular plates and the $\hat{r}$-$\hat{z}$ plane passing through the point of interest $\vec{P}$. The "top" plate is located a distance $\frac{1}{2}L$ above the $\hat{r}$-$\hat{y}$ and the "bottom" plate is located a distance $\frac{1}{2}L$ below. Of course, the radii of both plates is the same as the radius of the magnet, R.



Once again, we are going to look at very small element of area. But, this time, the elements of area are located on the two plates. The two distances we need to locate the center of any element of area are its distance r from the $\hat{z}$-axis and the angle ψ by which it is rotated around the $\hat{z}$-axis in the right-hand direction. Each element of area on the top plate has a corresponding element of area, with the same size, shape and location, on the bottom plate. We will examine the elements paired in this way.

The small elements of area are segments of annulus. The annulus in question has an inner radius of r and an outer radius of $r + \Delta r$, where Δr is the radial thickness of the annulus. The elements of the annulus are defined by the angle $\Delta\psi$ they subtend when viewed from the $\hat{z}$-axis. A detail of a typical small element of area is shown in the figure above.

The area ΔArea of a small element of area is given by:

$$\begin{aligned}\Delta\text{Area} &= (\text{annular thickness})(\text{fraction of circle}) \\ &= (\text{area at outer radius} - \text{area at inner radius})(\text{fraction of circle}) \\ &= [\pi(r+\Delta r)^2 - \pi r^2]\left(\frac{\Delta\psi}{2\pi}\right) \\ &= \pi\Delta r(2r+\Delta r)\left(\frac{\Delta\psi}{2\pi}\right) \\ &\cong r\Delta r\Delta\psi \text{ as } \Delta r \text{ gets very small} \qquad (32)\end{aligned}$$

We have now defined almost enough symbols for us to apply Dirac's Equation (30). Let us define σ^* as the density, per unit area, of the magnetic monopoles on the each plate. Then, the total magnetic charge on each small element of area is given by:

$$\begin{aligned} \text{magnetic charge} &= \Delta Area \times \sigma^* \\ &= \sigma^* r \Delta r \Delta\psi \qquad (33) \end{aligned}$$

As I said, we will be looking at the small elements of area in pairs, with the one on the top plate matched by a corresponding one on the bottom plate. The figure shows that vector $\overrightarrow{S_-}$ points from the origin to the small element of area on the bottom plate. To avoid cluttering the figure, the corresponding vector $\overrightarrow{S_+}$ from the origin to the small element of area on the top plate is not shown.

Now, vector $\overrightarrow{S_-P}$ points from the small element of area on the bottom plate to the point of interest. The corresponding vector $\overrightarrow{S_+P}$ from the small element of area on the top plate to the point of interest is not shown in the figure. As before, we will calculate these two vectors using vector subtraction.

The location of point $\vec{P}$ is the same as it was before, namely:

$$\vec{P} = P_r\hat{r} + P_z\hat{z} \qquad (18)$$

The components of the vectors $\overrightarrow{S_+}$ and $\overrightarrow{S_-}$ are given by:

$$\left.\begin{aligned} \overrightarrow{S_+} &= r\cos\psi\,\hat{r} + r\sin\psi\,\hat{y} + \tfrac{1}{2}L\hat{z} \\ \overrightarrow{S_-} &= r\cos\psi\,\hat{r} + r\sin\psi\,\hat{y} - \tfrac{1}{2}L\hat{z} \end{aligned}\right\} \qquad (34)$$

Using vector subtraction, the vectors $\overrightarrow{S_+P}$ and $\overrightarrow{S_-P}$ are given by:

$$\left.\begin{aligned} \overrightarrow{S_+P} &= \vec{P} - \overrightarrow{S_+} = (P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z - \tfrac{1}{2}L\right)\hat{z} \\ \overrightarrow{S_-P} &= \vec{P} - \overrightarrow{S_-} = (P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z + \tfrac{1}{2}L\right)\hat{z} \end{aligned}\right\} \qquad (35)$$

The square of the lengths of $\overrightarrow{S_+P}$ and $\overrightarrow{S_-P}$ are given by:

$$\left.\begin{aligned} \left|\overrightarrow{S_+P}\right|^2 &= (P_r - r\cos\psi)^2 + r^2\sin^2\psi + \left(P_z - \tfrac{1}{2}L\right)^2 \\ \left|\overrightarrow{S_-P}\right|^2 &= (P_r - r\cos\psi)^2 + r^2\sin^2\psi + \left(P_z + \tfrac{1}{2}L\right)^2 \end{aligned}\right\} \qquad (36)$$

Let us use the symbol $\overrightarrow{\Delta B_+}$ for the magnetic flux density generated by the element of area in the top plate. For this element, Equation (30) becomes:

$$\begin{aligned} \overrightarrow{\Delta B_+} &= \frac{1}{4\pi\mu_0}\frac{\text{magnetic charge}}{\left|\overrightarrow{S_+P}\right|^3}\overrightarrow{S_+P} \\ &= \frac{1}{4\pi\mu_0}\frac{\sigma^* r\Delta r\Delta\psi}{\left|\overrightarrow{S_+P}\right|^3}\overrightarrow{S_+P} \\ &= \frac{1}{4\pi\mu_0}\frac{\sigma^*\left[(P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z - \tfrac{1}{2}L\right)\hat{z}\right] r\Delta r\Delta\psi}{\left[(P_r - r\cos\psi)^2 + r^2\sin^2\psi + \left(P_z - \tfrac{1}{2}L\right)^2\right]^{3/2}} \qquad (37) \end{aligned}$$

In a similar way, the magnetic field $\overrightarrow{\Delta B_-}$ generated by the corresponding element of area in the bottom plate is given by (remembering that the magnetic charge on the bottom plate is algebraically negative):

$$\overrightarrow{\Delta B_-} \quad = -\frac{1}{4\pi\mu_0}\frac{\sigma^*\left[(P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z + \frac{1}{2}L\right)\hat{z}\right]r\Delta r\Delta\psi}{\left[(P_r - r\cos\psi)^2 + r^2\sin^2\psi + \left(P_z + \frac{1}{2}L\right)^2\right]^{3/2}} \quad (38)$$

The total flux density $\vec{B}$ at the point of interest $\vec{P}$ is found by adding up the contributions made by all the small elements of area in both the top and bottom plates. In the limit as the element of area is defined by distances small enough to be differentials, the summation becomes the following integral:

$$\vec{B} = \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{array}{l}\dfrac{(P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[(P_r - r\cos\psi)^2 + r^2\sin^2\psi + \left(P_z - \frac{1}{2}L\right)^2\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[(P_r - r\cos\psi)^2 + r^2\sin^2\psi + \left(P_z + \frac{1}{2}L\right)^2\right]^{3/2}}\end{array}\right\} r\,d\psi\,dr \quad (39)$$

The denominators can be expanded and a trigonometric identity used to re-write this expression as:

$$\vec{B} = \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{array}{l}\dfrac{(P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} - r\sin\psi\,\hat{y} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}}\end{array}\right\} r\,d\psi\,dr \quad (40)$$

The $\hat{y}$-component in each integral vanishes. Consider the traverse from $\psi = 0$ to $\psi = 2\pi$ for those elements of area which are a constant distance r from the $\hat{z}$-axis. The denominators are symmetric in ψ, because $\cos(-\psi) = \cos\psi$. The $\hat{y}$-term in the numerators is asymmetric, because $\sin(-\psi) = -\sin\psi$. The contributions to $\vec{B}$ from diametrically-opposing elements of area in a plate cancel each other out, with the result that the expression for $\vec{B}$ reduces to:

$$\vec{B}_{charge} = \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{array}{l}\dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}}\end{array}\right\} r\,d\psi\,dr \quad (41)$$

sing Special Case #1 to determine the surface monopole density σ^*

The expressions for the magnetic flux density found using the current model and the charge model are repeated here for comparison.

$$\vec{B}_{current} = \frac{\mu_0 J R}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + (R - P_r\cos\psi)\hat{z}}{\left[P_r^2 + R^2 + (P_z - z)^2 - 2RP_r\cos\psi\right]^{3/2}} d\psi dz \quad (27)$$

$$\vec{B}_{charge} = \frac{\mu_0 \sigma^*}{4\pi} \int_{r=0}^{r=R} \int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{l} \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} \end{array} \right\} r d\psi dr \quad (41)$$

Our objective is to show that these two expressions are equivalent. It would be very useful if the integrals could be taken in closed-form. While some progress can be made on the linear integrals in distances z and r, neither expression can be completely integrated in closed-form except in certain special cases. So, let us consider such a special case. In fact, we are going to use this special case to calculate the value of σ^* which makes the results identical. (The alert reader may notice that I have just stated an implicit conclusion: that the surface monopole charge density σ^* is constant across the faces. We used this as an implicit assumption in the derivation above in order to take σ^* outside of the summation / integral.)

The special case we will consider are points along the longitudinal axis of the magnet, being the $\hat{z}$-axis. For these points, we can set $P_r = 0$, which reduces the two integrals to:

$$\vec{B}_{current,\ on\ axis} = \frac{\mu_0 J R}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + R\hat{z}}{\left[R^2 + (P_z - z)^2\right]^{3/2}} d\psi dz \quad (42A)$$

$$\vec{B}_{charge,\ on\ axis} = \frac{\mu_0 \sigma^*}{4\pi} \int_{r=0}^{r=R} \int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{l} \dfrac{-r\cos\psi\,\hat{r} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[r^2 + \left(P_z - \frac{1}{2}L\right)^2\right]^{3/2}} + \cdots \\ \cdots - \dfrac{-r\cos\psi\,\hat{r} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[r^2 + \left(P_z + \frac{1}{2}L\right)^2\right]^{3/2}} \end{array} \right\} r d\psi dr \quad (42B)$$

In both cases, the radial component of $\vec{B}$ vanishes. The radial component of each integrand has a ψ-dependence of $\cos\psi$. For any particular value of r, the integral of $\cos\psi$ around a complete circle is zero. The axial integrands do not depend on angle ψ at all, so their integrals around the circle for any given value of r yields a factor of 2π. With the integral over ψ dealt with in this way, $\vec{B}_{current}$ and $\vec{B}_{charge}$ in this special case reduce to the following single integrals:

$$\vec{B}_{current,\ on\ axis} = \frac{\mu_0 JR}{2}\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{R}{\left[R^2+(z-P_z)^2\right]^{3/2}}\,dz\,\hat{z} \tag{43A}$$

$$\vec{B}_{charge,\ on\ axis} = \frac{\mu_0 \sigma^*}{2}\int_{r=0}^{r=R}\left\{\frac{\left(P_z-\frac{1}{2}L\right)}{\left[r^2+\left(P_z-\frac{1}{2}L\right)^2\right]^{3/2}}-\frac{\left(P_z+\frac{1}{2}L\right)}{\left[r^2+\left(P_z+\frac{1}{2}L\right)^2\right]^{3/2}}\right\}rdr\,\hat{z} \tag{43B}$$

Both of these integrals can be taken in closed-form. They can be solved using the following standard integrals and solutions, which are confirmed in Appendix "E":

$$\int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}+C \tag{E2}$$

$$\int \frac{xdx}{\left(x^2+a^2\right)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}+C \tag{E3}$$

Using Equation $(E2)$, we can write the integral for the current model as:

$$\begin{aligned}
\vec{B}_{current,\ on\ axis} &= \frac{\mu_0 JR}{2}\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{R}{\left[R^2+(z-P_z)^2\right]^{3/2}}\,dz\,\hat{z}\\
&= \frac{\mu_0 JR}{2}\left[\frac{1}{R^2}\frac{R(z-P_z)}{\sqrt{R^2+(z-P_z)^2}}\right]\Bigg|_{z=-\frac{L}{2}}^{z=+\frac{L}{2}}\hat{z}\\
&= \frac{\mu_0 J}{2}\left[\frac{\left(\frac{1}{2}L-P_z\right)}{\sqrt{R^2+\left(\frac{1}{2}L-P_z\right)^2}}+\frac{\left(\frac{1}{2}L+P_z\right)}{\sqrt{R^2+\left(\frac{1}{2}L+P_z\right)^2}}\right]\hat{z} \qquad (44A)
\end{aligned}$$

Similarly, using Equation $(E3)$ allows us to write the integral for the charge model as

$$\begin{aligned}
\vec{B}_{charge,\ on\ axis} &= -\frac{\mu_0 \sigma^*}{2}\left[\frac{\left(P_z-\frac{1}{2}L\right)}{\sqrt{r^2+\left(P_z-\frac{1}{2}L\right)^2}}-\frac{\left(P_z+\frac{1}{2}L\right)}{\sqrt{r^2+\left(P_z+\frac{1}{2}L\right)^2}}\right]\Bigg|_{r=0}^{r=R}\hat{z}\\
&= -\frac{\mu_0 \sigma^*}{2}\left[\begin{array}{c}\frac{\left(P_z-\frac{1}{2}L\right)}{\sqrt{R^2+\left(P_z-\frac{1}{2}L\right)^2}}-\frac{\left(P_z+\frac{1}{2}L\right)}{\sqrt{R^2+\left(P_z+\frac{1}{2}L\right)^2}}+\cdots\\ \cdots-\frac{\left(P_z-\frac{1}{2}L\right)}{\sqrt{\left(P_z-\frac{1}{2}L\right)^2}}+\frac{\left(P_z+\frac{1}{2}L\right)}{\sqrt{\left(P_z+\frac{1}{2}L\right)^2}}\end{array}\right]\hat{z}\\
&= \frac{\mu_0 \sigma^*}{2\mu_0}\left[\frac{\left(P_z+\frac{1}{2}L\right)}{\sqrt{R^2+\left(P_z+\frac{1}{2}L\right)^2}}-\frac{\left(P_z-\frac{1}{2}L\right)}{\sqrt{R^2+\left(P_z-\frac{1}{2}L\right)^2}}\right]\hat{z} \qquad (44B)
\end{aligned}$$

Note that the terms in the square brackets are the same. Then, the two equations are identical if we set the value of σ^* as follows:

$$\begin{aligned} & \vec{B}_{current,\ \ on\ axis} = \vec{B}_{charge,\ \ on\ axis} \\ \leftrightarrow \qquad & \frac{\mu_0 J}{2} = \frac{\mu_0 \sigma^*}{2} \\ \leftrightarrow \qquad & \sigma^* = J \qquad (45) \end{aligned}$$

Conclusion

The surface monopole model of a cylindrical permanent magnet gives the same magnetic flux density as the current sheet model if the charge density σ^* is set equal to the current sheet density J.

One additional thing to note is this: nothing in the mathematics above depends on the particular value of P_z. Physically, this means that nothing depends on the location of the point of interest along the longitudinal axis. In particular, the points of interest, at which the magnetic flux densities were calculated and compared, can be "inside" the magnet. Of course, in this special case, we have shown equivalence of the source models only on the longitudinal axis.

Special case #2: The far-field approximation

As a second special case, let us restrict our attention to points of interest (P_r, P_z) which are far away from the magnet. Because the point of interest is far away from the magnet, the approximate solutions for the magnetic flux density which result are called the "far-field" approximations. We can state this condition mathematically by saying that $\sqrt{P_r^2 + P_z^2}$ is much greater than both the integration variables r and z, or:

$$\frac{|r|}{\sqrt{P_r^2 + P_z^2}} \ll 1 \quad \text{and} \quad \frac{|z|}{\sqrt{P_r^2 + P_z^2}} \ll 1 \quad (46)$$

These assumptions simplify the denominators in the integrands of $\vec{B}_{current}$ and $\vec{B}_{charge}$. Three denominators are involved, one in Equation (27) for $\vec{B}_{current}$ and two in Equation (41) for $\vec{B}_{charge}$. All three denominators have the following generic form:

$$\begin{aligned} \text{Denominator} \quad &= \frac{1}{[P_r^2 + r^2 + (P_z - z)^2 - 2rP_r \cos\psi]^{3/2}} \\ &= \frac{1}{(P_r^2 + P_z^2)^{3/2}\left(1 + \frac{r^2 + z^2 - 2zP_z - 2rP_r \cos\psi}{P_r^2 + P_z^2}\right)^{3/2}} \\ &= \frac{1}{(P_r^2 + P_z^2)^{3/2}(1+\epsilon)^{3/2}} \qquad (47) \\ \text{where} \qquad \epsilon \quad &= \frac{r^2 + z^2 - 2zP_z - 2rP_r \cos\psi}{P_r^2 + P_z^2} \end{aligned}$$

The assumptions in Equation (46) mean that $\epsilon \ll 1$. In fact, the terms in r^2 and z^2 are an order even smaller than the terms in rP_r and zP_z. The MacLauren series expansion which can help us simplify this expression is:

$$(1+x)^{3/2} = 1 - \tfrac{3}{2}x + \left(\tfrac{1}{2!}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)x^2 + \mathcal{O}(x^3) \quad (48)$$

When x is very small, Equation (48) can be approximated by truncating the expansion and only keeping the desired number of terms. If we keep only one term, then:

$$(1+x)^{3/2} \cong 1 - \tfrac{3}{2}x \quad \text{when} \quad x \ll 1 \quad (49)$$

Applying this approximation to Equation (47) gives:

$$\begin{aligned}
\text{Denominator} \quad &= \frac{1}{(P_r^2+P_z^2)^{3/2}}\left\{1 - \frac{3}{2}\frac{r^2+z^2-2P_z z - 2rP_r\cos\psi}{P_r^2+P_z^2} + \mathcal{O}\left[\frac{r^2 \text{ and } z^2}{(P_r^2+P_z^2)^2}\right]\right\} \\
&= \frac{1}{(P_r^2+P_z^2)^{3/2}}\left\{1 + 3\frac{P_z z + r\cos\psi}{P_r^2+P_z^2} + \mathcal{O}\left[\frac{r^2 \text{ and } z^2}{(P_r^2+P_z^2)^2}\right]\right\} \\
&\cong \frac{1}{(P_r^2+P_z^2)^{3/2}}\left[1 + 3\frac{P_z z + r\cos\psi}{P_r^2+P_z^2}\right] \\
&= \frac{P_r^2+P_z^2+3P_z z+3rP_r\cos\psi}{(P_r^2+P_z^2)^{5/2}} \qquad (50)
\end{aligned}$$

Using Equation (50), Equation (27) for $\vec{B}_{current}$ can be approximated as follows:

$$\vec{B}_{current} = \frac{\mu_0 JR}{4\pi}\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}}\int_{\psi=0}^{\psi=2\pi}\frac{(P_z-z)\cos\psi\,\hat{r} + (R-P_r\cos\psi)\hat{z}}{[P_r^2+R^2+(P_z-z)^2-2RP_r\cos\psi]^{3/2}}\,d\psi dz \qquad (27)$$

$$\vec{B}_{current,\ far} \cong \frac{\mu_0 JR}{4\pi}\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{aligned}&\frac{(P_z-z)\cos\psi\,(P_r^2+P_z^2+3P_z z+3RP_r\cos\psi)\hat{r}}{(P_r^2+P_z^2)^{5/2}}+\cdots \\ &\cdots+\frac{(R-P_r\cos\psi)(P_r^2+P_z^2+3P_z z+3RP_r\cos\psi)\hat{z}}{(P_r^2+P_z^2)^{5/2}}\end{aligned}\right\}d\psi dz \qquad (51)$$

The numerators of the integrands include factors of $\cos\psi$ and $\cos^2\psi$. Collecting like terms gives:

$$\vec{B}_{current,\ far} = \frac{\mu_0 JR}{4\pi}\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{aligned}&\frac{\left[\begin{matrix}(P_z-z)(P_r^2+P_z^2+3P_z z)\cos\psi+\cdots \\ \cdots+3RP_r(P_z-z)\cos^2\psi\end{matrix}\right]\hat{r}}{(P_r^2+P_z^2)^{5/2}}+\cdots \\ &\cdots+\frac{\left[\begin{matrix}R(P_r^2+P_z^2+3P_z z+3RP_r\cos\psi)+\cdots \\ \cdots-P_r\cos\psi\,(P_r^2+P_z^2+3P_z z+3RP_r\cos\psi)\end{matrix}\right]\hat{z}}{(P_r^2+P_z^2)^{5/2}}\end{aligned}\right\}d\psi dz \qquad (52)$$

The terms whose ψ-dependence is $\cos\psi$ will vanish upon integration around the circle from $\psi = 0$ to $\psi = 0$. Foreseeing this, we can simplify Equation (52) to:

$$\vec{B}_{current,\ far} = \frac{\mu_0 JR}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{c} \dfrac{3RP_r(P_z - z)\cos^2\psi\,\hat{r}}{(P_r^2 + P_z^2)^{5/2}} + \cdots \\ \cdots + \dfrac{[R(P_r^2 + P_z^2 + 3P_z z) - 3RP_r^2\cos^2\psi]\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \end{array} \right\} d\psi dz \quad (53)$$

The two dependencies on ψ which remain are easily integrated. When the integrand is a constant with respect to ψ, the integral will be equal to 2π. When the integrand depends on $\cos^2\psi$, the integral over ψ will be equal to π. Equation (53) then becomes:

$$\begin{aligned} \vec{B}_{current,\ far} &= \frac{\mu_0 JR}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \left\{ \frac{3\pi RP_r(P_z - z)\hat{r}}{(P_r^2 + P_z^2)^{5/2}} + \frac{[2\pi R(P_r^2 + P_z^2 + 3P_z z) - 3\pi RP_r^2]\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right\} dz \\ &= \frac{\mu_0 JR^2}{4} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \left[\frac{3P_r(P_z - z)\hat{r}}{(P_r^2 + P_z^2)^{5/2}} + \frac{(-P_r^2 + 2P_z^2 + 6P_z z)\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right] dz \end{aligned} \quad (54)$$

The remaining integrals, over z, are easily taken.

$$\begin{aligned} \vec{B}_{current,\ far} &= \frac{\mu_0 JR^2}{4} \left[\frac{3P_r\left(P_z z - \frac{1}{2}z^2\right)\hat{r}}{(P_r^2 + P_z^2)^{5/2}} + \frac{(-P_r^2 z + 2P_z^2 z + 3P_z z^2)\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right] \Bigg|_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \\ &= \frac{\mu_0 JR^2}{4} \left[\frac{3P_r P_z z\hat{r}}{(P_r^2 + P_z^2)^{5/2}} + \frac{(-P_r^2 + 2P_z^2) z\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right] \Bigg|_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \\ &= \frac{\mu_0 JR^2 L}{4} \left[\frac{3P_r P_z\hat{r}}{(P_r^2 + P_z^2)^{5/2}} + \frac{(-P_r^2 + 2P_z^2)\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right] \\ &= \frac{\mu_0 JR^2 L}{4} \left[\frac{3P_r P_z\hat{r} + (2P_z^2 - P_r^2)\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right] \end{aligned} \quad (55)$$

I have some comments to make about this but, before I do, let us process $\vec{B}_{charge}$ under the far field assumptions and show that it comes out the same. Equation (41) for $\vec{B}_{charge}$ has two denominators with the generic form in Equation (47), one with the value of z fixed at $z = +\frac{1}{2}L$ and the other with the value of z fixed at $z = -\frac{1}{2}L$. We can therefore approximate Equation (41) as follows:

$$\vec{B}_{charge} = \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{matrix}\dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}}\end{matrix}\right\} r\,d\psi\,dr \qquad (41)$$

$$\vec{B}_{charge,\ far} \cong \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{matrix}\dfrac{(P_r - r\cos\psi)\left(P_r^2 + P_z^2 + \frac{3}{2}P_z L + 3rP_r\cos\psi\right)\hat{r}}{\left(P_r^2 + P_z^2\right)^{5/2}} + \cdots \\ \cdots + \dfrac{\left(P_z - \frac{1}{2}L\right)\left(P_r^2 + P_z^2 + \frac{3}{2}P_z L + 3rP_r\cos\psi\right)\hat{z}}{\left(P_r^2 + P_z^2\right)^{5/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\left(P_r^2 + P_z^2 - \frac{3}{2}P_z L + 3rP_r\cos\psi\right)\hat{r}}{\left(P_r^2 + P_z^2\right)^{5/2}} + \cdots \\ \cdots - \dfrac{\left(P_z + \frac{1}{2}L\right)\left(P_r^2 + P_z^2 - \frac{3}{2}P_z L + 3rP_r\cos\psi\right)\hat{z}}{\left(P_r^2 + P_z^2\right)^{5/2}}\end{matrix}\right\} r\,d\psi\,dr \qquad (56)$$

A great many of the terms in the curly brackets cancel each other out. When the algebra is finished, what remains is:

$$\vec{B}_{charge,\ far} = \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\frac{L\begin{bmatrix}3P_z(P_r - r\cos\psi)\hat{r} + \cdots \\ \cdots + (2P_z^2 - P_r^2 - 3rP_r\cos\psi)\hat{z}\end{bmatrix}}{\left(P_r^2 + P_z^2\right)^{5/2}} r\,d\psi\,dr \qquad (57)$$

Two of the terms in the numerator have $\cos\psi$ as a factor. That is their only dependence on ψ. When we integrate around the circle from $\psi = 0$ to $\psi = 0$, these two terms will vanish. Foreseeing this, we can simplify Equation (57) to:

$$\vec{B}_{charge,\ far} = \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\int_{\psi=0}^{\psi=2\pi}\frac{L[3P_z P_r\hat{r} + (2P_z^2 - P_r^2)\hat{z}]}{\left(P_r^2 + P_z^2\right)^{5/2}} r\,d\psi\,dr \qquad (58)$$

The integrand has no further dependence on variable ψ, so the integral around the circle can be taken first, to give:

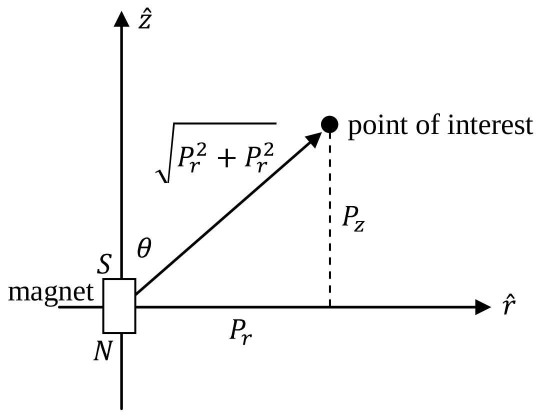
$$\begin{aligned}\vec{B}_{charge,\ far} &= \frac{\mu_0\sigma^*}{4\pi}\int_{r=0}^{r=R}\frac{L[3P_z P_r\hat{r} + (2P_z^2 - P_r^2)\hat{z}]}{\left(P_r^2 + P_z^2\right)^{5/2}}\left\{\int_{\psi=0}^{\psi=2\pi} d\psi\right\} r\,dr \\ &= \frac{\mu_0\sigma^*}{2}\int_{r=0}^{r=R}\frac{L[3P_z P_r\hat{r} + (2P_z^2 - P_r^2)\hat{z}]}{\left(P_r^2 + P_z^2\right)^{5/2}} r\,dr \qquad (59)\end{aligned}$$

Because the fraction is a constant with respect to the integration variable r, it can be taken outside the integral and the integration completed as follows:

$$
\begin{aligned}
\vec{B}_{charge,\ far} &= \frac{\mu_0 \sigma^*}{2} \frac{L[3P_z P_r \hat{r} + (2P_z^2 - P_r^2)\hat{z}]}{(P_r^2 + P_z^2)^{5/2}} \int_{r=0}^{r=R} r dr \\
&= \frac{\mu_0 \sigma^* R^2 L}{4} \left[\frac{3P_z P_r \hat{r} + (2P_z^2 - P_r^2)\hat{z}}{(P_r^2 + P_z^2)^{5/2}} \right] \\
&= \vec{B}_{current,\ far} \quad \text{if } \sigma^* = J
\end{aligned} \tag{60}
$$

So, we have shown that, as one gets further and further from the magnet, the magnetic flux densities predicted by the two models get closer and closer. (This is not to say that the two models make different predictions near the magnet – in fact, they do – only that our mathematical proof in this section is limited to points of interest which are suitably far away.)

The term in square brackets can be interpreted geometrically, as shown in the following figure.



The plane shown is any radial plane which includes the axis of the magnet. The magnet itself is shown in cross-section as the rectangle centered on the origin. The point of interest, which is highlighted by the heavy dot, is a relatively long distance away compared with the dimensions of the magnet. It is customary to define the angle θ as the angle between the axis of the magnet (as defined by the vector from the north face to the south face) and the ray to the point of interest. The co-ordinates of the point of interest are related to this angle by:

$$
\left.
\begin{aligned}
\sin\theta &= \frac{P_r}{\sqrt{P_r^2 + P_z^2}} \\
\cos\theta &= \frac{P_z}{\sqrt{P_r^2 + P_z^2}}
\end{aligned}
\right\} \tag{61}
$$

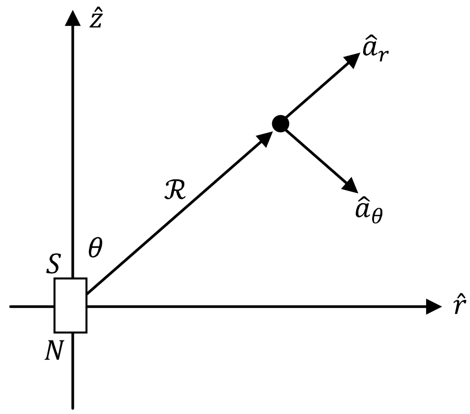
A bit of algebra can be used to show that:

$$
\left.
\begin{aligned}
\frac{3P_z P_r}{(P_r^2 + P_z^2)^{5/2}} &= \frac{3\sin\theta\cos\theta}{(P_r^2 + P_z^2)^{3/2}} \\
\frac{2P_z^2 - P_r^2}{(P_r^2 + P_z^2)^{5/2}} &= \frac{2\cos^2\theta - \sin^2\theta}{(P_r^2 + P_z^2)^{3/2}} = \frac{3\cos^2\theta - 1}{(P_r^2 + P_z^2)^{3/2}}
\end{aligned}
\right\} \tag{62}
$$

If we use the symbol $\mathcal{R}$ for the distance from the origin to the point of interest, being $\sqrt{P_r^2 + P_z^2}$, then the far field approximation for the magnetic flux density can be written as:

$$\vec{B}_{far} = \left(\frac{\mu_0 J R^2}{4}\right)\frac{3\sin\theta\cos\theta\,\hat{r} + (3\cos^2\theta - 1)\hat{z}}{\mathcal{R}^3} \quad (63)$$

This is the common expression for the far field, when expressed in cylindrical co-ordinates. The expression is frequently written in spherical co-ordinates as well. The axes for spherical co-ordinates are shown in the following figure.



Just so there is no confusion between the $\hat{r}$-$\hat{z}$ cylindrical co-ordinate frame in which we have been working, and the new spherical frame, I have labeled the axes $\hat{a}_r$ and $\hat{a}_\theta$ in the latter. The components of the magnetic field intensity are readily transformed. For the radial component in spherical co-ordinates, we have:

$$\begin{aligned}
\vec{B}_{far}\big|_{\hat{a}_r} &= \sin\theta\,\vec{B}_{far}\big|_{\hat{r}} + \cos\theta\,\vec{B}_{far}\big|_{\hat{z}} \\
&= \left(\frac{\mu_0\sigma^* L R^2}{4}\right)\left(\sin\theta\frac{3\sin\theta\cos\theta}{\mathcal{R}^3} + \cos\theta\frac{3\cos^2\theta - 1}{\mathcal{R}^3}\right) \\
&= \left(\frac{\mu_0\sigma^* L R^2}{4}\right)\left(\frac{\cos\theta}{\mathcal{R}^3}\right)(3\sin^2\theta + 3\cos^2\theta - 1) \\
&= \left(\frac{\mu_0\sigma^* L R^2}{4}\right)\left(\frac{2\cos\theta}{\mathcal{R}^3}\right) \qquad (64A)
\end{aligned}$$

and, for the polar angle component, we have:

$$\begin{aligned}
\vec{B}_{far}\big|_{\hat{a}_\theta} &= \cos\theta\,\vec{B}_{far}\big|_{\hat{r}} - \sin\theta\,\vec{B}_{far}\big|_{\hat{z}} \\
&= \left(\frac{\mu_0\sigma^* L R^2}{4}\right)\left(\cos\theta\frac{3\sin\theta\cos\theta}{\mathcal{R}^3} - \sin\theta\frac{3\cos^2\theta - 1}{\mathcal{R}^3}\right) \\
&= \left(\frac{\mu_0\sigma^* L R^2}{4}\right)\left(\frac{\sin\theta}{\mathcal{R}^3}\right)(3\cos^2\theta - 3\cos^2\theta + 1) \\
&= \left(\frac{\mu_0\sigma^* L R^2}{4}\right)\left(\frac{\sin\theta}{\mathcal{R}^3}\right) \qquad (64B)
\end{aligned}$$

In its complete spherical form, the magnetic flux density is:

$$\vec{B}_{charge} = \left(\frac{\mu_0 \sigma^* L R^2}{4}\right)\left(\frac{1}{\mathcal{R}^3}\right)(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta) \quad (65)$$

Special case #3: The central plane

As a third special case, let us restrict our attention to points of interest (P_r, P_z) which are located on the plane which bisects the magnet in the longitudinal direction. This plane passes through the geometric center of the magnet. For these points, $P_z = 0$ and Equations (27) and (41) reduce to:

$$\vec{B}_{current,\ central} = \frac{\mu_0 J R}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{-z\cos\psi\,\hat{r} + (R - P_r\cos\psi)\hat{z}}{(P_r^2 + R^2 + z^2 - 2RP_r\cos\psi)^{3/2}} d\psi dz \quad (66A)$$

$$\vec{B}_{charge,\ central} = \frac{\mu_0 \sigma^*}{4\pi} \int_{r=0}^{r=R} \int_{\psi=0}^{\psi=2\pi} \frac{-L\hat{z}}{\left(P_r^2 + r^2 + \frac{1}{4}L^2 - 2P_r r\cos\psi\right)^{3/2}} r d\psi dr \quad (66B)$$

Note that, for the charge model, the radial component of the flux density vanishes immediately. It should be the case, therefore, that the radial component for the current model is also zero. To see that it is, let us take the integral in the $\hat{z}$-direction first. When ψ is held fixed, the pertinent integral is:

$$\begin{aligned}
\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{-z\cos\psi}{(P_r^2 + R^2 + z^2 - 2RP_r\cos\psi)^{3/2}} dz &= \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{-\cos\psi\, dz^2}{(P_r^2 + R^2 + z^2 - 2RP_r\cos\psi)^{3/2}} \\
&= \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{-\cos\psi\, d(P_r^2 + R^2 + z^2 - 2RP_r\cos\psi)}{(P_r^2 + R^2 + z^2 - 2RP_r\cos\psi)^{3/2}} \\
&= \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} 2\cos\psi\, d\left(\frac{1}{\sqrt{P_r^2 + R^2 + z^2 - 2RP_r\cos\psi}}\right) \\
&= \left(\frac{2\cos\psi}{\sqrt{P_r^2 + R^2 + z^2 - 2RP_r\cos\psi}}\right)\Bigg|_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \\
&= 0 \qquad (67)
\end{aligned}$$

So, the two source models agree that the radial component of the flux density is zero on the central plane. That leaves us to deal with the axial component. The axial component of both of Equations (66) can be integrated over their linear variables, z in the case of $\vec{B}_{current,\ central}$ and r in the case of $\vec{B}_{charge,\ central}$. In fact, we can do better than this. Even in the general cases of Equations (27) and (41), for $\vec{B}_{current}$ and $\vec{B}_{charge}$, both the radial and axial components can be integrated over their linear variables. There are four integrals, which are described in Appendices "A" through "D" attached hereto. Because of the complexity of the integrals, I will not repeat the results here. The interested reader can consult the appendices to find:

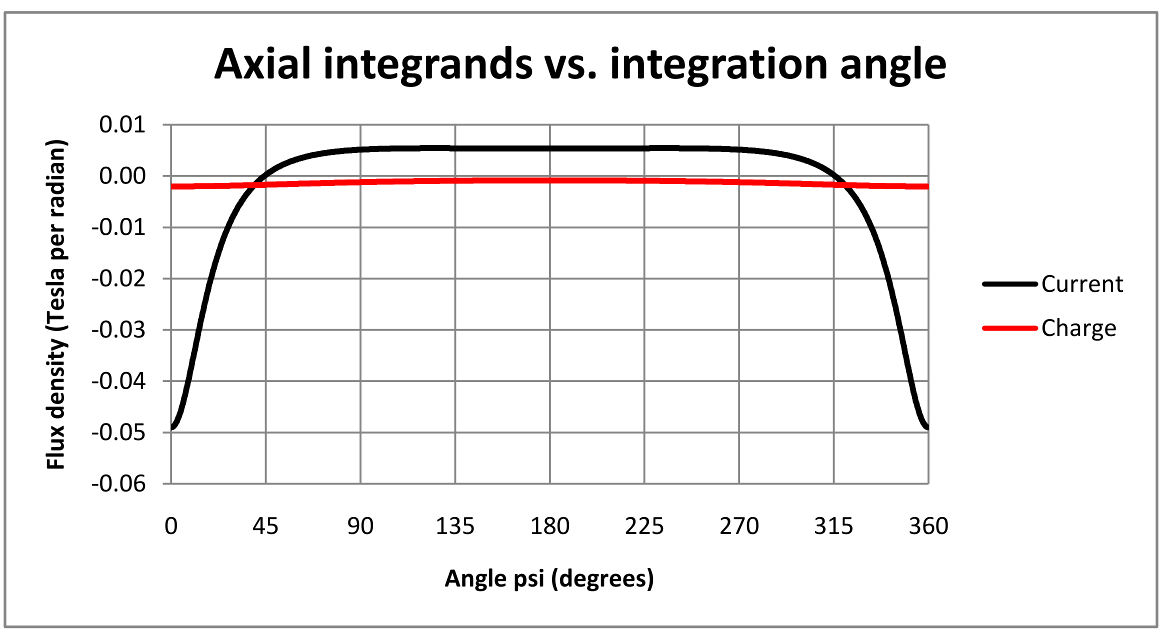
Appendix "A"	Radial component of $\vec{B}_{current}$
Appendix "B"	Radial component of $\vec{B}_{charge}$
Appendix "C"	Axial component of $\vec{B}_{current}$
Appendix "D"	Axial component of $\vec{B}_{charge}$

For the special case at hand, being the central plane, we have already shown that there is no radial component in either model. Taking the general result from Appendices "C" and "D", and substituting $P_z = 0$ for this special case, we have:

$$\vec{B}_{current,\ central} = \frac{\mu_0 J}{4\pi}\int_{\psi=0}^{\psi=2\pi}\left(\frac{R - P_r\cos\psi}{P_r^2 + R^2 - 2RP_r\cos\psi}\right)\frac{RL}{\sqrt{P_r^2 + R^2 + \frac{1}{4}L^2 - 2RP_r\cos\psi}}\,d\psi\,\hat{z} \qquad (C3')$$

$$\vec{B}_{charge,\ central} = \frac{\mu_0\sigma^*}{4\pi}\int_{\psi=0}^{\psi=2\pi}\left\{\begin{array}{c}\dfrac{L\left(P_r^2 + \frac{1}{4}L^2 - P_r R\cos\psi\right)}{\left(P_r^2 + \frac{1}{4}L^2 - P_r^2\cos^2\psi\right)\sqrt{P_r^2 + R^2 + \frac{1}{4}L^2 - 2P_r R\cos\psi}} + \cdots \\ \cdots - \dfrac{L\sqrt{P_r^2 + \frac{1}{4}L^2}}{\left(P_r^2 + \frac{1}{4}L^2 - P_r^2\cos^2\psi\right)}\end{array}\right\} d\psi\,\hat{z} \qquad (D4')$$

Note that the two integrals are single integrals, over the angle ψ around a circle. It is certainly not obvious that the two integrals are the same. It is not even obvious that the integrands have the same values for corresponding values of ψ. Indeed, they do not. In order to show this, let us consider a numerical example using a magnet which is two centimeters long ($L = 0.02$ m) and one centimeter in diameter ($R = 0.005$ m). The particular values of J and σ^* do not really matter so long as $\sigma^* = J$. For the graph plotted, I used $J = 10^6$ A/m. I used a point of interest which is in the central plane and two millimeters from the surface, that is, $P_z = 0$ and $P_r = 0.007$ m. The following graph shows the magnitude of the integrands [including the constant factor which is outside of the integral in Equations $(C3')$ and $(D4')$] for values of ψ around a circle.



Do not fuss too much about the physical interpretation of this graph, because they really is not any. What is shown is the value of the integrands with respect to the variable of integration ψ. All that is physically realizable is the evaluated integral, being the area under the two curves. Both integrals have the same value: $B_z = -0.008383$ Tesla. The minus sign indicates that the magnetic flux at this point points in the minus $\hat{z}$-direction, which is entirely consistent with having positively-charged magnetic monopoles on the top face and negatively-charged magnetic monopoles on the bottom face. It is also entirely consistent with the direction chosen for the current, which gives rise to flux density inside the magnet which points in the positive $\hat{z}$-direction.

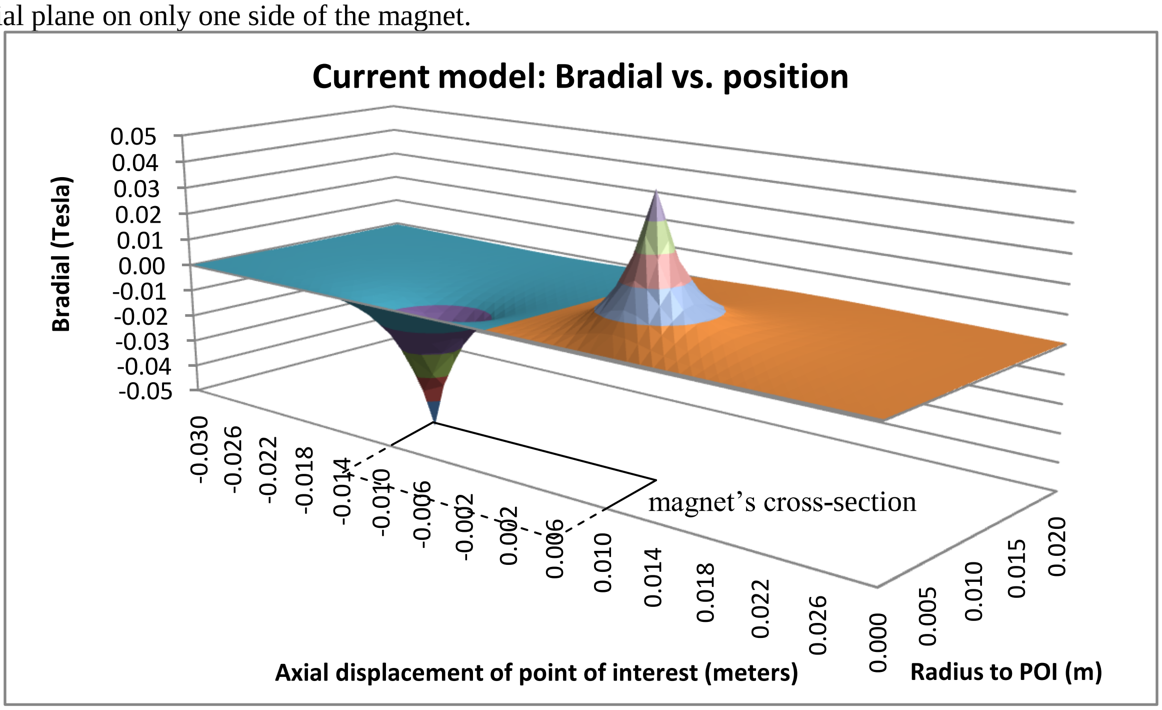
One thing which the graph makes instantly clear is that the integrands in Equations $(C3')$ and $(D4')$ are not the same for the same values of ψ. It follows that a direct attack – trying to show equality of the two integrals by showing equality of the integrands – will fail. Furthermore, the graphs shows that both curves, and the difference between them, are symmetric around the variable of integration. Integrating the difference between the curves from $\psi = 0$ to $\psi = 2\pi$ will give a zero result if the integrals of the separate curves are the same but, because of the symmetry, we cannot rely on the cancellation of terms to help us through the integration.

In fact, neither Equations $(C3')$ and $(D4')$ nor any combination of them can be integrated in closed form. They are forms of elliptic integrals, which come in different kinds, all of them intractable. Remember, too, that I announced this section as a "special case". The central plane is indeed a special case but we cannot prove equally between the two source models even for it.

Graphically showing the equality for a particular magnet

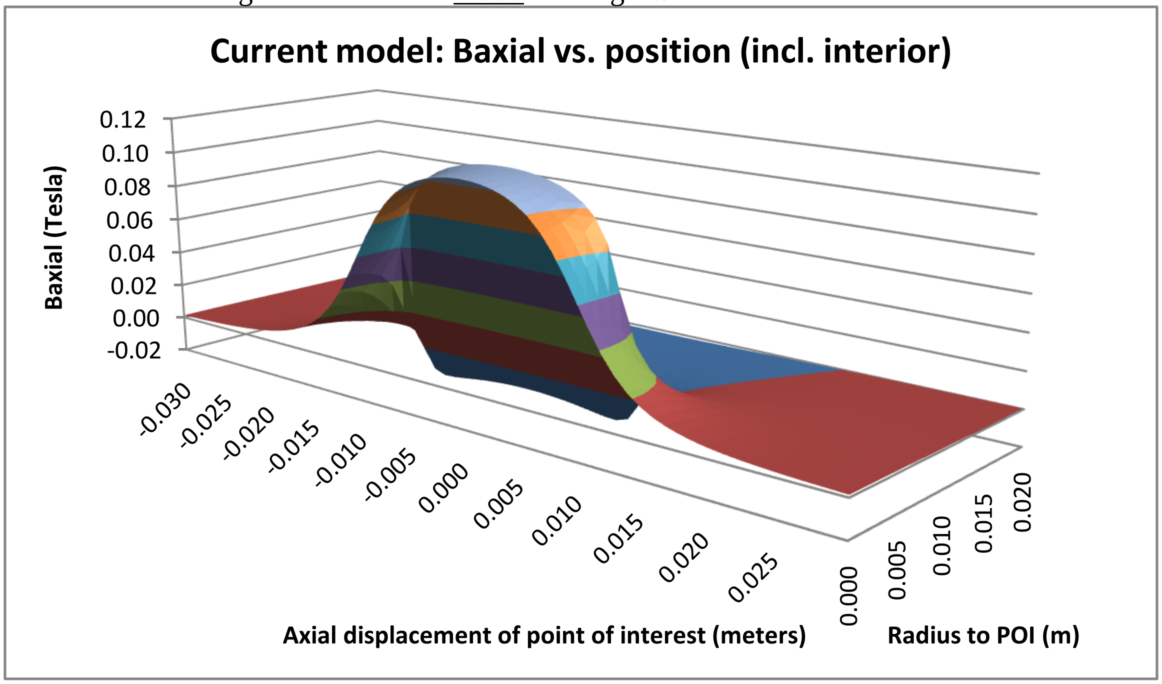
For arbitrary points of interest, located at P_r and P_z, there is no prospect of closed-form solutions to Equations (27) and (41) and no prospect of mathematically showing that they produce the same values for the magnetic flux density. The two equations can, however, be easily plotted for any given permanent magnet. The following graphs are plots of the flux density of a cylindrical permanent magnet which is two centimeters long and one centimeter in diameter, with a lineal current density of $J = 100{,}000$ A/m.

The first plot shows the radial component of the flux density inside and around the magnet, as calculated using the current sheet source model. The horizontal axis pointing towards the left represents the central axis of the magnet. Off-axis distances increase towards the right. What is shown is the flux density in a radial plane on only one side of the magnet.



The radial component reaches its peak at the "rims" of the magnet, along the circles where the end-faces intersect the cylindrical sides. This is where the lines of flux emanating from the face "turn the corner", if you will, to begin their traverse towards the opposing face. The closer the lines of flux are to the rims, the more sharply the turn the corner.

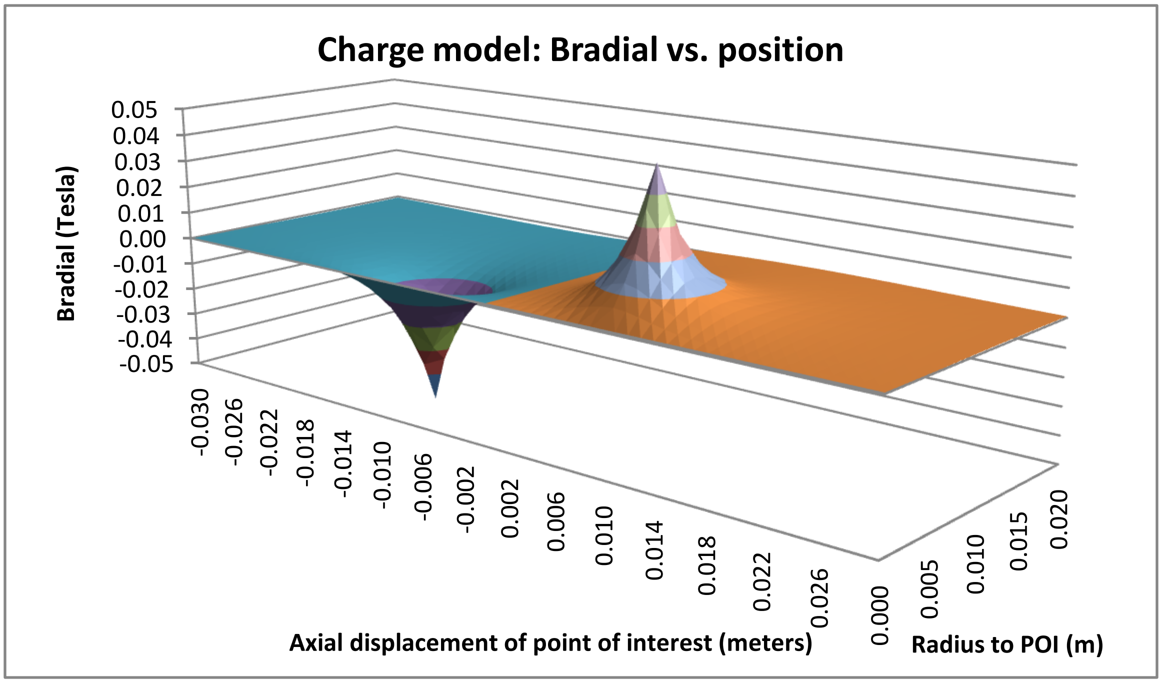
The next plot shows the axial component of the flux density produced by the current sheet source model. Once again, the flux density is shown inside the magnet as well as outside. I will have something to say below about the meanings of these values inside the magnet.



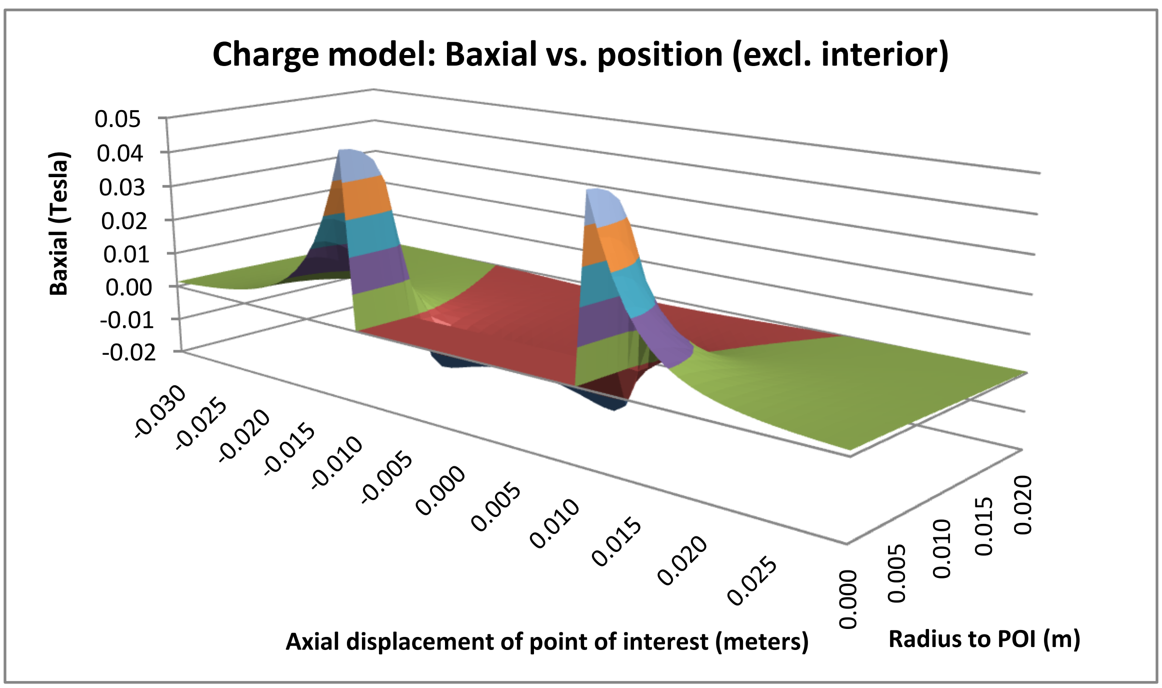
The axial component reaches its maximum value at the center of the magnet and decreases through the faces. In the region outside the curved surface of the magnet, the axial component is algebraically negative. It is in this region that the lines of flux close their path from one face to the other. The magnet being examined here is not very long compared with its diameter. If its length-to-diameter ratio was bigger, we would notice that the axial component of the flux density was quite uniform inside the magnet, both along its axis and across its diameters.

Of course, the interior of a permanent magnet is not physically accessible, so the magnetic flux density there is not really relevant. However, the interior is accessible and relevant in a wire-wound solenoid. In circumstances where one models a wire-wound solenoid as a current sheet, the values shown obtain.

The following plot shows the radial component of the magnetic flux density calculated using the surface charge model, both outside and inside the permanent magnet. It is identical to the plot calculated using the current sheet model. Despite some authoritative assertions to the contrary, the charge model does correctly predict the radial component of the field in the interior of a solenoid.



But, the authorities are correct about the axial component of the flux density produced by the charge model – the charge model does not give the same flux density as the current sheet model at points inside the magnet. Outside the magnet, it does. The following plot shows the axial component from the charge model, <u>for the outside of the magnet only</u>.



It is not very easy to see that this plot is exactly the same as the plot for the axial component of the current sheet model, in the region outside the magnet. To make that clear, the following plot shows the current sheet's plot once again, but with the interior of the magnet zeroed out.

These plots were produced using a short Visual basic program, whose code is listed in Appendix “F” attached hereto.

We can make the practical conclusion, even through we cannot prove it mathematically, that the current sheet source model and the surface monopole charge source model predict the same magnetic flux density at all points outside of a cylindrical permanent magnet with uniform magnetization along its axis. The two representations are the same if we set the monopole charge density on the end faces equal to the current sheet current density around the curved surface. Equations (27) and (41) are the expressions to use to calculate the flux densities of the current model and the charge model, respectively.

What I have not done in this paper is move on to the next step to show how the source models can be used. Interested readers can follow up on this topic in the paper titled *Forces and torques between cylindrical magnets and ideal solenoids*.

Jim Hawley
November 2012

An e-mail setting out errors and omissions would be appreciated.

Appendix "A"

Integrating the radial component of $\vec{B}_{current}$ in the axial direction

The magnetic field described by the current source model is:

$$\vec{B}_{current} = \frac{\mu_0 JR}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + (R - P_r\cos\psi)\hat{z}}{[P_r^2 + R^2 + (P_z - z)^2 - 2RP_r\cos\psi]^{3/2}} d\psi dz \qquad (6)$$

The radial component is:

$$\vec{B}_{current}\Big|_{\hat{r}} = \frac{\mu_0 JR}{4\pi} \int_{\psi=0}^{\psi=2\pi} \mathcal{I}\cos\psi\, d\psi$$

$$\text{where} \quad \mathcal{I} = -\int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{(z - P_z)}{[P_r^2 + R^2 + (z - P_z)^2 - 2RP_r\cos\psi]^{3/2}} dz \qquad (A1)$$

$\mathcal{I}$ can be integrated with respect to the variable z using the following standard integral, which is confirmed in Appendix "E":

$$\int \frac{xdx}{(x^2 + a^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}} + C \qquad (E3)$$

Then:

$$\begin{aligned} \mathcal{I} &= \left[\frac{1}{\sqrt{P_r^2 + R^2 + (z - P_z)^2 - 2RP_r\cos\psi}}\right]\Bigg|_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \\ &= \left[\begin{array}{c} \dfrac{1}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} - P_z\right)^2 - 2RP_r\cos\psi}} + \cdots \\ -\dfrac{1}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} + P_z\right)^2 - 2RP_r\cos\psi}} \end{array}\right] \qquad (A2) \end{aligned}$$

so that:

$$\vec{B}_{current}\Big|_{\hat{r}} = \frac{\mu_0 J}{4\pi} \int_{\psi=0}^{\psi=2\pi} \left[\begin{array}{c} \dfrac{R\cos\psi}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} - P_z\right)^2 - 2RP_r\cos\psi}} + \cdots \\ -\dfrac{R\cos\psi}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} + P_z\right)^2 - 2RP_r\cos\psi}} \end{array}\right] d\psi \qquad (A3)$$

Appendix "B"

Integrating the radial component of $\vec{B}_{charge}$ in the radial direction

The magnetic field described by the surface charge model is:

$$\vec{B}_{charge} = \frac{\mu_0 \sigma^*}{4\pi} \int_{r=0}^{r=R} \int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{l} \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} \end{array} \right\} r\,d\psi\,dr \qquad (4)$$

The radial component is:

$$\vec{B}_{charge}\Big|_{\hat{r}} = \frac{\mu_0 \sigma^*}{4\pi} \int_{\psi=0}^{\psi=2\pi} (\mathcal{I} - \mathcal{J})\,d\psi$$

$$\text{where} \quad \mathcal{I} = \int_{r=0}^{r=R} \frac{(P_r - r\cos\psi)}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} r\,dr \qquad (B1)$$

$$\text{and} \quad \mathcal{J} = \int_{r=0}^{r=R} \frac{(P_r - r\cos\psi)}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} r\,dr \qquad (B2)$$

$\mathcal{J}$ has exactly the same form of dependence on r as does $\mathcal{I}$, so the same form of solution will obtain. $\mathcal{I}$ can be integrated with respect to the variable r using the following two standard integrals, which are confirmed in Appendix "E":
:

$$\int \frac{x\,dx}{(x^2 + bx + c)^{3/2}} = -\frac{2bx + 4c}{(4c - b^2)\sqrt{x^2 + bx + c}} + C \qquad (E7)$$

$$\int \frac{x^2\,dx}{(x^2 + bx + c)^{3/2}} = \left[\begin{array}{l} \ln\left(2\sqrt{x^2 + bx + c} + 2x + b\right) + \cdots \\ \cdots - \dfrac{(4c - 2b^2)x - 2bc}{(4c - b^2)\sqrt{x^2 + bx + c}} + C \end{array} \right] \qquad (E8)$$

Then:

$$\mathcal{I} = \left\{ \begin{array}{l} -P_r \dfrac{-P_r r\cos\psi + P_r^2 + \left(P_z - \frac{1}{2}L\right)^2}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi}} + \cdots \\ \cdots - \cos\psi \ln\left(2\sqrt{P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi} + 2r - 2P_r\cos\psi\right) + \cdots \\ \cdots + \cos\psi \dfrac{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r^2\cos^2\psi\right] r + P_r\cos\psi\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2\right]}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi}} \end{array} \right\} \Bigg|_{r=0}^{r=R}$$

and, continuing,

$$\mathcal{J} = \left\{ \begin{array}{c} \dfrac{\left[\!\left[\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2\right](-P_r + P_r\cos^2\psi + r\cos\psi) + \cdots \atop \cdots + P_r^2 r\cos\psi\,(1 - 2\cos^2\psi) \right]\!\right]}{\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - 2P_r r\cos\psi}} + \cdots \\ \cdots - \cos\psi\ln\left(2\sqrt{P_r^2 + r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - 2P_r r\cos\psi} + 2r - 2P_r\cos\psi\right) \end{array} \right\}\Bigg|_{r=0}^{r=R}$$

$$= \left\{ \begin{array}{c} \dfrac{\left[\!\left[\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2\right](-P_r + P_r\cos^2\psi + R\cos\psi) + \cdots \atop \cdots + P_r^2 R\cos\psi\,(1 - 2\cos^2\psi) \right]\!\right]}{\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + R^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - 2P_r R\cos\psi}} + \cdots \\ \cdots - \cos\psi\ln\left(2\sqrt{P_r^2 + R^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - 2P_r R\cos\psi} + 2R - 2P_r\cos\psi\right) + \cdots \\ \cdots + \dfrac{P_r(1 - \cos^2\psi)\sqrt{P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2}}{\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]} + \cdots \\ \cdots + \cos\psi\ln\left(2\sqrt{P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2} - 2P_r\cos\psi\right) \end{array} \right\} \quad (B3)$$

so that:

$$\vec{B}_{charge}\Big|_{\hat{r}} = \frac{\mu_0\sigma^*}{4\pi}\int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{c} \dfrac{\left[\!\left[\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2\right](-P_r + P_r\cos^2\psi + R\cos\psi) + \cdots \atop \cdots + P_r^2 R\cos\psi\,(1 - 2\cos^2\psi) \right]\!\right]}{\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + R^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - 2P_r R\cos\psi}} + \cdots \\ \cdots - \dfrac{\left[\!\left[\left[P_r^2 + \left(P_z + \tfrac{1}{2}L\right)^2\right](-P_r + P_r\cos^2\psi + R\cos\psi) + \cdots \atop \cdots + P_r^2 R\cos\psi\,(1 - 2\cos^2\psi) \right]\!\right]}{\left[P_r^2 + \left(P_z + \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + R^2 + \left(P_z + \tfrac{1}{2}L\right)^2 - 2P_r R\cos\psi}} + \cdots \\ \cdots - \cos\psi\ln\left(\dfrac{\sqrt{P_r^2 + R^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - 2P_r R\cos\psi} + R - P_r\cos\psi}{\sqrt{P_r^2 + R^2 + \left(P_z + \tfrac{1}{2}L\right)^2 - 2P_r R\cos\psi} + R - P_r\cos\psi}\right) + \cdots \\ \cdots + \cos\psi\ln\left(\dfrac{\sqrt{P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2} - P_r\cos\psi}{\sqrt{P_r^2 + \left(P_z + \tfrac{1}{2}L\right)^2} - P_r\cos\psi}\right) + \cdots \\ \cdots + \dfrac{P_r(1 - \cos^2\psi)\sqrt{P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2}}{\left[P_r^2 + \left(P_z - \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]} - \dfrac{P_r(1 - \cos^2\psi)\sqrt{P_r^2 + \left(P_z + \tfrac{1}{2}L\right)^2}}{\left[P_r^2 + \left(P_z + \tfrac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]} \end{array} \right\} d\psi \quad (B4)$$

Appendix "C"

Integrating the axial component of $\vec{B}_{current}$ in the axial direction

The magnetic field described by the current source model is:

$$\vec{B}_{current} = \frac{\mu_0 JR}{4\pi} \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \int_{\psi=0}^{\psi=2\pi} \frac{(P_z - z)\cos\psi\,\hat{r} + (R - P_r\cos\psi)\hat{z}}{[P_r^2 + R^2 + (P_z - z)^2 - 2RP_r\cos\psi]^{3/2}} d\psi dz \qquad (6)$$

The axial component is:

$$\vec{B}_{current}\Big|_{\hat{z}} = \frac{\mu_0 JR}{4\pi} \int_{\psi=0}^{\psi=2\pi} \mathcal{I}(R - P_r\cos\psi) d\psi$$

$$\text{where} \quad \mathcal{I} = \int_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \frac{1}{[P_r^2 + R^2 + (z - P_z)^2 - 2RP_r\cos\psi]^{3/2}} dz \qquad (C1)$$

$\mathcal{I}$ can be integrated with respect to the variable z using the following standard integral, which is confirmed in Appendix "E":

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{x^2 + a^2}} + C \qquad (E6)$$

Then:

$$\begin{aligned} \mathcal{I} &= \left[\frac{1}{P_r^2 + R^2 - 2RP_r\cos\psi}\frac{(z - P_z)}{\sqrt{P_r^2 + R^2 + (z - P_z)^2 - 2RP_r\cos\psi}}\right]\Bigg|_{z=-\frac{L}{2}}^{z=+\frac{L}{2}} \\ &= \frac{1}{P_r^2 + R^2 - 2RP_r\cos\psi}\left[\begin{array}{l} \dfrac{\left(\frac{L}{2} - P_z\right)}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} - P_z\right)^2 - 2RP_r\cos\psi}} + \cdots \\ \cdots + \dfrac{\left(\frac{L}{2} + P_z\right)}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} + P_z\right)^2 - 2RP_r\cos\psi}} \end{array}\right] \end{aligned} \qquad (C2)$$

so that:

$$\vec{B}_{current}\Big|_{\hat{z}} = \frac{\mu_0 J}{4\pi} \int_{\psi=0}^{\psi=2\pi} R\left(\frac{R - P_r\cos\psi}{P_r^2 + R^2 - 2RP_r\cos\psi}\right)\left[\begin{array}{l} \dfrac{\left(\frac{L}{2} - P_z\right)}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} - P_z\right)^2 - 2RP_r\cos\psi}} + \cdots \\ \cdots + \dfrac{\left(\frac{L}{2} + P_z\right)}{\sqrt{P_r^2 + R^2 + \left(\frac{L}{2} + P_z\right)^2 - 2RP_r\cos\psi}} \end{array}\right] d\psi \qquad (C3)$$

Appendix "D"

Integrating the axial component of $\vec{B}_{charge}$ in the radial direction

The magnetic field described by the surface charge model is as follows:

$$\vec{B}_{charge} = \frac{\mu_0 \sigma^*}{4\pi} \int_{r=0}^{r=R} \int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{l} \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z - \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} + \cdots \\ \cdots - \dfrac{(P_r - r\cos\psi)\hat{r} + \left(P_z + \frac{1}{2}L\right)\hat{z}}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} \end{array} \right\} r\,d\psi\,dr \qquad (4)$$

The axial component is:

$$\vec{B}_{charge}\Big|_{\hat{z}} = \frac{\mu_0 \sigma^*}{4\pi} \int_{\psi=0}^{\psi=2\pi} \left[\left(P_z - \tfrac{1}{2}L\right)\mathcal{I} - \left(P_z + \tfrac{1}{2}L\right)\mathcal{J}\right] d\psi$$

$$\text{where} \quad \mathcal{I} = \int_{r=0}^{r=R} \frac{1}{\left[P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} r\,dr \qquad (D1)$$

$$\text{and} \quad \mathcal{J} = \int_{r=0}^{r=R} \frac{1}{\left[P_r^2 + r^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r r\cos\psi\right]^{3/2}} r\,dr \qquad (D2)$$

$\mathcal{J}$ has exactly the same form of dependence on r as does $\mathcal{I}$, so the same form of solution will obtain. $\mathcal{I}$ can be integrated with respect to the variable r using the following standard integral, which is confirmed in Appendix "E":

$$\int \frac{x\,dx}{(x^2 + bx + c)^{3/2}} = -\frac{2bx + 4c}{(4c - b^2)\sqrt{x^2 + bx + c}} + C \qquad (E7)$$

Then:

$$\mathcal{I} = \left\{ -\frac{-P_r r\cos\psi + P_r^2 + \left(P_z - \frac{1}{2}L\right)^2}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + r^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r r\cos\psi}} \right\}\Bigg|_{r=0}^{r=R}$$

$$= \left\{ \begin{array}{l} -\dfrac{-P_r R\cos\psi + P_r^2 + \left(P_z - \frac{1}{2}L\right)^2}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + R^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r R\cos\psi}} + \cdots \\ \cdots + \dfrac{P_r^2 + \left(P_z - \frac{1}{2}L\right)^2}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2\cos^2\psi\right]\sqrt{P_r^2 + \left(P_z - \frac{1}{2}L\right)^2}} \end{array} \right\} \qquad (D3)$$

so that:

$$\vec{B}_{charge}\Big|_{\hat{z}} = \frac{\mu_0 \sigma^*}{4\pi} \int_{\psi=0}^{\psi=2\pi} \left\{ \begin{array}{c} -\dfrac{\left(P_z - \frac{1}{2}L\right)\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r R \cos\psi\right]}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2 \cos^2\psi\right]\sqrt{P_r^2 + R^2 + \left(P_z - \frac{1}{2}L\right)^2 - 2P_r R \cos\psi}} + \cdots \\ \cdots + \dfrac{\left(P_z - \frac{1}{2}L\right)\sqrt{P_r^2 + \left(P_z - \frac{1}{2}L\right)^2}}{\left[P_r^2 + \left(P_z - \frac{1}{2}L\right)^2 - P_r^2 \cos^2\psi\right]} + \cdots \\ \cdots + \dfrac{\left(P_z + \frac{1}{2}L\right)\left[P_r^2 + \left(P_z + \frac{1}{2}L\right)^2 - P_r R \cos\psi\right]}{\left[P_r^2 + \left(P_z + \frac{1}{2}L\right)^2 - P_r^2 \cos^2\psi\right]\sqrt{P_r^2 + R^2 + \left(P_z + \frac{1}{2}L\right)^2 - 2P_r R \cos\psi}} + \cdots \\ \cdots - \dfrac{\left(P_z + \frac{1}{2}L\right)\sqrt{P_r^2 + \left(P_z + \frac{1}{2}L\right)^2}}{\left[P_r^2 + \left(P_z + \frac{1}{2}L\right)^2 - P_r^2 \cos^2\psi\right]} \end{array} \right\} d\psi \qquad (D4)$$

Appendix "E"

Selected elliptic integrals in standard form

(where a, b and c are real constants)

1. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+C \qquad (E1)$$

2. $$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}+C \qquad (E2)$$

3. $$\int \frac{xdx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}+C \qquad (E3)$$

4. $$\int \frac{dx}{\sqrt{x^2+bx+c}} = \ln\left(2\sqrt{x^2+bx+c}+2x+b\right)+C \qquad (E4)$$

5. $$\int \frac{xdx}{\sqrt{x^2+bx+c}} = \begin{bmatrix} \sqrt{x^2+bx+c}+\cdots \\ \cdots-\frac{1}{2}b\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)+C \end{bmatrix} \qquad (E5)$$

6. $$\int \frac{dx}{(x^2+bx+c)^{3/2}} = \frac{4x+2b}{(4c-b^2)\sqrt{x^2+bx+c}}+C \qquad (E6)$$

7. $$\int \frac{xdx}{(x^2+bx+c)^{3/2}} = -\frac{2bx+4c}{(4c-b^2)\sqrt{x^2+bx+c}}+C \qquad (E7)$$

8. $$\int \frac{x^2dx}{(x^2+bx+c)^{3/2}} = \begin{bmatrix} \ln\left(2\sqrt{x^2+bx+c}+2x+b\right)+\cdots \\ \cdots-\frac{(4c-2b^2)x-2bc}{(4c-b^2)\sqrt{x^2+bx+c}}+C \end{bmatrix} \qquad (E8)$$

Confirmation of 1.

$$\frac{d}{dx}\left[\ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)\right] = \frac{\frac{d}{dx}\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)}{\frac{x+\sqrt{x^2+a^2}}{a}}$$

$$= \frac{1+\frac{d}{dx}\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}$$

$$= \frac{1+\frac{\left(\frac{1}{2}\right)(2x)}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}$$

$$= \frac{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}$$

$$= \frac{1}{\sqrt{x^2+a^2}} \qquad (E1')$$

Confirmation of 2.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}\right) &= \frac{1}{a^2}\left[\frac{1}{\sqrt{x^2+a^2}}+\frac{x\left(-\frac{1}{2}\right)(2x)}{\left(x^2+a^2\right)^{3/2}}\right]\\ &= \frac{1}{a^2}\left[\frac{x^2+a^2}{\left(x^2+a^2\right)^{3/2}}+\frac{-x^2}{\left(x^2+a^2\right)^{3/2}}\right]\\ &= \frac{1}{\left(x^2+a^2\right)^{3/2}} \qquad (E2')\end{aligned}$$

Confirmation of 3.

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{\sqrt{x^2+a^2}}\right) &= -\frac{\left(-\frac{1}{2}\right)(2x)}{\left(x^2+a^2\right)^{3/2}}\\ &= \frac{x}{\left(x^2+a^2\right)^{3/2}} \qquad (E3')\end{aligned}$$

Confirmation of 4.

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)\right] &= \frac{\frac{d}{dx}\left(2\sqrt{x^2+bx+c}+2x+b\right)}{2\sqrt{x^2+bx+c}+2x+b}\\ &= \frac{2\frac{\left(\frac{1}{2}\right)(2x+b)}{\sqrt{x^2+bx+c}}+2}{2\sqrt{x^2+bx+c}+2x+b}\\ &= \frac{2x+b+2\sqrt{x^2+bx+c}}{\sqrt{x^2+bx+c}\left(2\sqrt{x^2+bx+c}+2x+b\right)}\\ &= \frac{1}{\sqrt{x^2+bx+c}} \qquad (E4')\end{aligned}$$

Confirmation of 5.

$$\frac{d}{dx}\left[\begin{array}{c}\sqrt{x^2+bx+c}+\cdots\\ \cdots-\frac{1}{2}b\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)\end{array}\right] = \left\{\begin{array}{c}\frac{\left(\frac{1}{2}\right)(2x+b)}{\sqrt{x^2+bx+c}}+\cdots\\ \cdots-\frac{1}{2}b\frac{d}{dx}\left[\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)\right]\end{array}\right\}$$

and, using $(E4')$,

$$\begin{aligned}&= \frac{\left(\frac{1}{2}\right)(2x+b)}{\sqrt{x^2+bx+c}}-\frac{1}{2}b\frac{1}{\sqrt{x^2+bx+c}},\\ &= \frac{x}{\sqrt{x^2+bx+c}} \qquad (E5')\end{aligned}$$

Confirmation of 6.

$$\frac{d}{dx}\left[\frac{4x+2b}{(4c-b^2)\sqrt{x^2+bx+c}}\right] = \left[\begin{array}{c}\dfrac{4}{(4c-b^2)\sqrt{x^2+bx+c}}+\cdots \\ \cdots+\dfrac{(4x+2b)\left(-\frac{1}{2}\right)(2x+b)}{(4c-b^2)(x^2+bx+c)^{3/2}}\end{array}\right]$$

$$= \frac{\left[\begin{array}{c}4(x^2+bx+c)+\cdots \\ \cdots-(2x+b)(2x+b)\end{array}\right]}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$= \frac{(4c-b^2)}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$= \frac{1}{(x^2+bx+c)^{3/2}} \qquad (E6')$$

Confirmation of 7.

$$\frac{d}{dx}\left[-\frac{2bx+4c}{(4c-b^2)\sqrt{x^2+bx+c}}\right] = \left[\begin{array}{c}-\dfrac{2b}{(4c-b^2)\sqrt{x^2+bx+c}}+\cdots \\ \cdots-\dfrac{(2bx+4c)\left(-\frac{1}{2}\right)(2x+b)}{(4c-b^2)(x^2+bx+c)^{3/2}}\end{array}\right]$$

$$= \frac{\left[\begin{array}{c}-2b(x^2+bx+c)+\cdots \\ \cdots-(2bx+4c)\left(-\frac{1}{2}\right)(2x+b)\end{array}\right]}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$= \frac{-b^2x+4cx}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$= \frac{x}{(x^2+bx+c)^{3/2}} \qquad (E7')$$

Confirmation of 8.

$$\frac{d}{dx}\left[\begin{array}{c}\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)+\cdots \\ \cdots-\dfrac{(4c-2b^2)x-2bc}{(4c-b^2)\sqrt{x^2+bx+c}}\end{array}\right] = \left\{\begin{array}{c}\dfrac{d}{dx}\left[\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)\right]+\cdots \\ \cdots-\dfrac{(4c-2b^2)}{(4c-b^2)\sqrt{x^2+bx+c}}+\cdots \\ \cdots-\dfrac{[(4c-2b^2)x-2bc]\left(-\frac{1}{2}\right)(2x+b)}{(4c-b^2)(x^2+bx+c)^{3/2}}\end{array}\right\}$$

and, using $(E4')$,

$$= \left[\begin{array}{c}\dfrac{1}{\sqrt{x^2+bx+c}}+\cdots \\ \cdots-\dfrac{(4c-2b^2)}{(4c-b^2)\sqrt{x^2+bx+c}}+\cdots \\ \cdots+\dfrac{(2cx-b^2x-bc)(2x+b)}{(4c-b^2)(x^2+bx+c)^{3/2}}\end{array}\right]$$

and, continuing,

$$\frac{d}{dx}\left[\begin{array}{c}\ln\left(2\sqrt{x^2+bx+c}+2x+b\right)+\cdots\\ \cdots-\dfrac{(4c-2b^2)x-2bc}{(4c-b^2)\sqrt{x^2+bx+c}}\end{array}\right]=\frac{\left[\begin{array}{c}(4c-b^2)(x^2+bx+c)+\cdots\\ \cdots-(4c-2b^2)(x^2+bx+c)+\cdots\\ \cdots+(2cx-b^2x-bc)(2x+b)\end{array}\right]}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$=\frac{\left(\begin{array}{c}b^2x^2+b^3x+b^2c+\cdots\\ \cdots+4cx^2+2bcx-2b^2x^2-b^3x-2bcx-b^2c\end{array}\right)}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$=\frac{4cx^2-b^2x^2}{(4c-b^2)(x^2+bx+c)^{3/2}}$$

$$=\frac{x^2}{(x^2+bx+c)^{3/2}} \qquad (E8')$$

Appendix "F"

Listing of the Visual Basic program

The program was developed in Visual Basic 2010 Express. It has a single Form and no GUI. Parameters are specified in the code. The results are written to an Excel file where they can be plotted.

```
Option Strict On
Option Explicit On

' Permanent Magnet - Summation1
' Calculates the components of the magnetic field around a cylindrical magnet, using
' both the current sheet model and the surface charge model.

Public Class Form1
    Inherits System.Windows.Forms.Form

    ' Dimensions of the magnet.
    Public Lmagnet As Double = 0.02
    Public Rmagnet As Double = 0.005

    ' Physical properties.
    Public Mu0 As Double = 4 * Math.PI * 0.0000001
    Public J As Double = 100000
    Public Sigma As Double = J

    ' The point of interest (Pr, Pz).
    Public Pz As Double
    Public PZstart As Double = -0.03
    Public PZstop As Double = +0.03
    Public NumPZ As Int32 = 61
    Public DelPZ As Double = (PZstop - PZstart) / (NumPZ - 1)
    Public Pr As Double
    Public PRstart As Double = 0
    Public PRstop As Double = +0.02
    Public NumPR As Int32 = 21
    Public DelPR As Double = (PRstop - PRstart) / (NumPR - 1)

    ' Integration variables.
    Public Z As Double
    Public NumZ As Int32 = 1000
    Public DelZ As Double = Lmagnet / 1000
    Public Psi As Double
    Public NumPsi As Int32 = 1000
    Public DelPsi As Double = 2 * Math.PI / NumPsi
    Public R As Double
    Public NumR As Int32 = 1000
    Public DelR As Double = Rmagnet / 1000

    ' Magnetic field contributions from a single volume or current element.
    Public Br1E As Double
    Public Bz1E As Double

    ' Total magnetic field at the point of interest.
    Public Br As Double
    Public Bz As Double
    Public Btotal As Double
```

```
' Variables relating to the Excel file.
' Add COM object "Microsoft Excel 12.0 Object Library" to project references.
Public objExcel As Microsoft.Office.Interop.Excel.Application
Public objExcelWB As Microsoft.Office.Interop.Excel.Workbook
Public objExcelWS As Microsoft.Office.Interop.Excel.Worksheet
Public ExcelFileName As String = "C:\MagnetSummation1.xlsx"

Public Sub New()
    InitializeComponent()
    With Me
        Name = ""
        Text = "Permanent Magnet - Summation #1"
        FormBorderStyle = Windows.Forms.FormBorderStyle.FixedSingle
        Size = New Drawing.Size(800, 720)
        CenterToScreen()
        Visible = True
        Controls.Add(labelDisplay) : labelDisplay.BringToFront()
        Controls.Add(buttonExit) : buttonExit.BringToFront()
        Controls.Add(buttonStartExecution) : buttonStartExecution.BringToFront()
        PerformLayout()
    End With
End Sub

' Controls and handlers.

Public labelDisplay As New Windows.Forms.Label With _
    {.Size = New Drawing.Size(200, 600), _
     .Location = New Drawing.Point(5, 40), _
     .Text = "", .TextAlign = ContentAlignment.TopLeft}

Public WithEvents buttonExit As New Windows.Forms.Button With _
{.Size = New Drawing.Size(200, 30), _
 .Location = New Drawing.Point(210, 5), _
 .Text = "Quit and exit", .TextAlign = ContentAlignment.MiddleCenter}

Public Sub buttonExit_Click() Handles buttonExit.MouseClick
    Application.Exit()
End Sub

Public WithEvents buttonStartExecution As New Windows.Forms.Button With _
    {.Size = New Drawing.Size(200, 30), _
     .Location = New Drawing.Point(5, 5), _
     .Text = "Start execution", .TextAlign = ContentAlignment.MiddleCenter}

Public Sub buttonStartExecution_Click() Handles buttonStartExecution.MouseClick
    RunProgram()
End Sub

Public Sub RunProgram()
    '
    ' Open the Excel file for output.
    Try
        objExcel = CType(CreateObject("Excel.Application"),  _
            Microsoft.Office.Interop.Excel.Application)
        objExcel.Visible = False
        objExcelWB = CType(objExcel.Workbooks.Open(ExcelFileName),  _
            Microsoft.Office.Interop.Excel.Workbook)
```

```
    objExcelWS = CType(objExcelWB.Sheets("Sheet1"),  _
        Microsoft.Office.Interop.Excel.Worksheet)
Catch ex As Exception
    Cursor.Current = Cursors.Default
    MsgBox("Could not open the Excel file.", vbOKOnly)
    Exit Sub
End Try
'
' Write the run information to the Excel file.
With objExcelWS
    objExcelWS.Cells(1, 1) = "Permanent Magnet - Summation #1"
    objExcelWS.Cells(2, 1) = "Lmagnet = " & Trim(Str(Lmagnet)) & "m"
    objExcelWS.Cells(3, 1) = "Rmagnet = " & Trim(Str(Rmagnet)) & "m"
    objExcelWS.Cells(4, 1) = "PZstart = " & Trim(Str(PZstart)) & "m"
    objExcelWS.Cells(5, 1) = "PZstop = " & Trim(Str(PZstop)) & "m"
    objExcelWS.Cells(6, 1) = "DelPZ = " & Trim(Str(DelPZ)) & "m"
    objExcelWS.Cells(7, 1) = "NumPZ = " & Trim(Str(NumPZ))
    objExcelWS.Cells(8, 1) = "PRstart = " & Trim(Str(PRstart)) & "m"
    objExcelWS.Cells(9, 1) = "PRstop = " & Trim(Str(PRstop)) & "m"
    objExcelWS.Cells(10, 1) = "DelPR = " & Trim(Str(DelPR)) & "m"
    objExcelWS.Cells(11, 1) = "NumPR = " & Trim(Str(NumPR))
    objExcelWS.Cells(12, 1) = "NumZ = " & Trim(Str(NumZ))
    objExcelWS.Cells(13, 1) = "NumPsi = " & Trim(Str(NumPsi))

    objExcelWS.Cells(15, 1) = "r-component of B - Current Model"
    objExcelWS.Cells(16, 1) = "Pr (m)->"
    objExcelWS.Cells(17, 1) = "Pz (m)"
    objExcelWS.Cells(15 + 5 + NumPZ, 1) = "z-component of B - Current Model"
    objExcelWS.Cells(16 + 5 + NumPZ, 1) = "Pr (m)->"
    objExcelWS.Cells(17 + 5 + NumPZ, 1) = "Pz (m)"

    objExcelWS.Cells(15, 1 + 3 + NumPR) = "r-component of B - Charge Model"
    objExcelWS.Cells(16, 1 + 3 + NumPR) = "Pr (m)->"
    objExcelWS.Cells(17, 1 + 3 + NumPR) = "Pz (m)"
    objExcelWS.Cells(15 + 5 + NumPZ, 1 + 3 + NumPR) = _
        "z component of B - Charge Model"
    objExcelWS.Cells(16 + 5 + NumPZ, 1 + 3 + NumPR) = "Pr (m)->"
    objExcelWS.Cells(17 + 5 + NumPZ, 1 + 3 + NumPR) = "Pz (m)"
End With
'
'/////////////////////////////////////////////////////
'// Current Model
'/////////////////////////////////////////////////////
'
' Main loop to step through the vertical displacements Pz.
For IPz As Int32 = 1 To NumPZ Step 1
    Pz = PZstart + ((IPz - 1) * DelPZ)
    '
    ' Write the Pz headers for two tables.
    objExcelWS.Cells(17 + IPz, 1) = Trim(Str(Pz))
    objExcelWS.Cells(17 + 5 + NumPZ + IPz, 1) = Trim(Str(Pz))
    '
    ' Main loop to step through the radial displacements Pr.
    For IPr As Int32 = 1 To NumPR Step 1
        Pr = PRstart + ((IPr - 1) * DelPR)
        '
        ' Write the Pr headers for two tables.
        objExcelWS.Cells(16, 1 + IPr) = Trim(Str(Pr))
```

```
        objExcelWS.Cells(16 + 5 + NumPZ, 1 + IPr) = Trim(Str(Pr))
        '
        ' Set up for the integration.
        Br = 0
        Bz = 0
        '
        ' Sub-loop to integrate around the circle.
        Dim CosPsi As Double
        Dim Temp1 As Double
        Dim Temp2 As Double
        Dim Temp3 As Double
        For Ipsi As Int32 = 1 To NumPsi Step 1
            Psi = (Ipsi - 0.5) * DelPsi
            CosPsi = Math.Cos(Psi)
            '
            ' Sub-loop to integrate up the length.
            For Iz As Int32 = 1 To NumZ Step 1
                Z = (-Lmagnet / 2) + ((Iz - 0.5) * DelZ)
                '
                ' Do the calculations.
                Temp1 = Pz - Z
                Temp2 = (Pr * Pr) + (Rmagnet * Rmagnet) + (Temp1 * Temp1) + _
                    (-2 * Rmagnet * Pr * CosPsi)
                Temp2 = Temp2 ^ 1.5
                Temp3 = Rmagnet - (Pr * CosPsi)
                Br1E = Temp1 * CosPsi / Temp2
                Bz1E = Temp3 / Temp2
                Br = Br + Br1E
                Bz = Bz + Bz1E
            Next Iz
        Next Ipsi
        Br = Mu0 * J * Rmagnet * Br * DelPsi * DelZ / (4 * Math.PI)
        Bz = Mu0 * J * Rmagnet * Bz * DelPsi * DelZ / (4 * Math.PI)
        '
        ' Write the magnetic field components for the Current Model.
        objExcelWS.Cells(17 + IPz, 1 + IPr) = Trim(Str(Br))
        objExcelWS.Cells(17 + 5 + NumPZ + IPz, 1 + IPr) = Trim(Str(Bz))
        '
        ' Display progress on the monitor.
        labelDisplay.Text = "Now calculating Current Model:" & vbCrLf & _
            "Pr = " & Trim(Str(Pr)) & " m" & vbCrLf & _
            "Pz = " & Trim(Str(Pz)) & " m"
        labelDisplay.Refresh()
        ' Give other processes a chance to work.
        Application.DoEvents()
    Next IPr
Next IPz
'
'/////////////////////////////////////////////////////
'// Charge Model
'/////////////////////////////////////////////////////
'
' Main loop to step through the vertical displacements Pz.
For IPz As Int32 = 1 To NumPZ Step 1
    Pz = PZstart + ((IPz - 1) * DelPZ)
    '
    ' Write the Pz headers for two tables.
    objExcelWS.Cells(17 + IPz, 1 + 3 + NumPR) = Trim(Str(Pz))
```

```
objExcelWS.Cells(17 + 5 + NumPZ + IPz, 1 + 3 + NumPR) = Trim(Str(Pz))
'
' Main loop to step through the radial displacements Pr.
For IPr As Int32 = 1 To NumPR Step 1
    Pr = PRstart + ((IPr - 1) * DelPR)
    '
    ' Write the Pr headers for two tables.
    objExcelWS.Cells(16, 1 + 3 + NumPR + IPr) = Trim(Str(Pr))
    objExcelWS.Cells(16 + 5 + NumPZ, 1 + 3 + NumPR + IPr) = Trim(Str(Pr))
    '
    ' Set up for the integration.
    Br = 0
    Bz = 0
    '
    ' Sub-loop to integrate around the circle.
    Dim CosPsi As Double
    Dim Temp1 As Double
    Dim Temp2 As Double
    Dim Temp3 As Double
    Dim Temp4 As Double
    Dim Temp5 As Double
    For Ipsi As Int32 = 1 To NumPsi Step 1
        Psi = (Ipsi - 0.5) * DelPsi
        CosPsi = Math.Cos(Psi)
        '
        ' Sub-loop to integrate out the radius.
        Dim DelArea As Double
        Dim Rinner As Double
        Dim Router As Double
        For Ir As Int32 = 1 To NumR Step 1
            Rinner = (Ir - 1) * DelR
            Router = Ir * DelR
            DelArea = 0.5 * ((Router * Router) - (Rinner * Rinner)) * DelPsi
            If (Ir = 1) Then
                R = 0
            Else
                R = ((Ir - 0.5) * DelR)
            End If
            '
            ' Do the calculations.
            Temp1 = Pr - (R * CosPsi)
            Temp2 = Pz - (Lmagnet / 2)
            Temp3 = Pz + (Lmagnet / 2)
            Temp4 = (Pr * Pr) + (R * R) + (Temp2 * Temp2) + _
                (-2 * Pr * R * CosPsi)
            Temp4 = Temp4 ^ 1.5
            Temp5 = (Pr * Pr) + (R * R) + (Temp3 * Temp3) + _
                (-2 * Pr * R * CosPsi)
            Temp5 = Temp5 ^ 1.5
            Br1E = (Temp1 / Temp4) - (Temp1 / Temp5)
            Bz1E = (Temp2 / Temp4) - (Temp3 / Temp5)
            Br1E = Br1E * DelArea
            Bz1E = Bz1E * DelArea
            Br = Br + Br1E
            Bz = Bz + Bz1E
        Next Ir
    Next Ipsi
    Br = Mu0 * Sigma * Br / (4 * Math.PI)
```

```
                Bz = Mu0 * Sigma * Bz / (4 * Math.PI)
                '
                ' Write the magnetic field components for the Charge Model.
                objExcelWS.Cells(17 + IPz, 1 + 3 + NumPR + IPr) = Trim(Str(Br))
                objExcelWS.Cells(17 + 5 + NumPZ + IPz, 1 + 3 + NumPR + IPr) = _
                    Trim(Str(Bz))
                '
                ' Display progress on the monitor.
                labelDisplay.Text = "Now calculating Charge Model:" & vbCrLf & _
                    "Pr = " & Trim(Str(Pr)) & " m" & vbCrLf & _
                    "Pz = " & Trim(Str(Pz)) & " m"
                labelDisplay.Refresh()
                ' Give other processes a chance to work.
                Application.DoEvents()
            Next IPr
        Next IPz
        ' Save the Excel file.
        objExcelWB.Save()
        objExcelWB.Close()
        MsgBox("All done.")
    End Sub

End Class
```