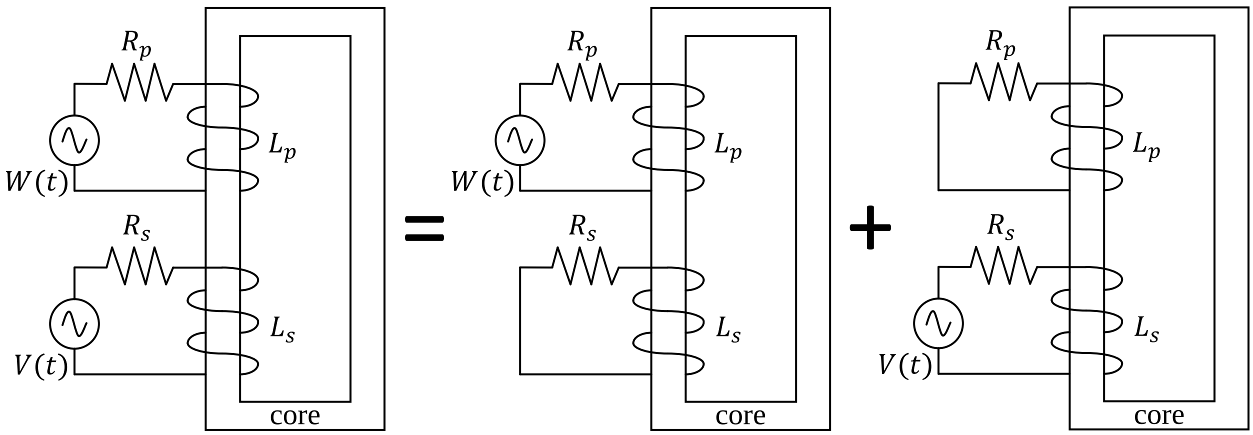
The ideal transformer with an active secondary circuit

Reference is made to a previous paper: *The ideal transformer with a single secondary winding*. This paper extends that analysis to the case where the secondary winding has its own independent voltage source.

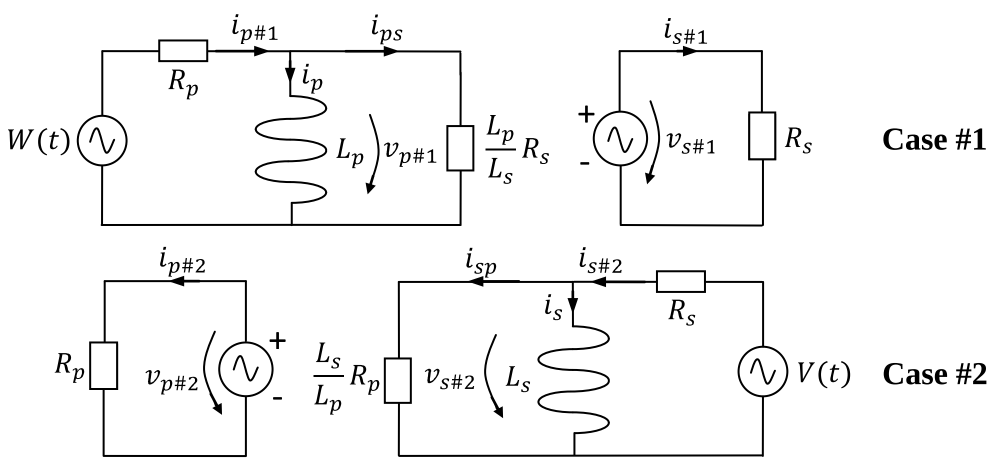
The analysis is based on the principle of superposition, as illustrated in the following diagram.



What we will do is this: we will analyze the circuit twice, first with only one of the voltage sources included, and then again with only the other voltage source included. Then, we will add the results.

In the figure, R_p is the resistance in the primary winding, including the lumped resistance of inductor L_p. Similarly, the load R_s in the secondary circuit includes the equivalent series resistance of the secondary winding L_s. As before, we will consider the transformer to be ideal in all other respects.

The equivalent circuit diagrams for the two cases are as follows:



Some care has been taken to use symbols in such a way that there is no repetition. Furthermore, the directions of the currents and voltages have been chosen to be consistent with basic model of an ideal transformer as described in *The ideal transformer with a single secondary winding*. However, this does mean that we have to watch out for the algebraic signs of the currents when we add them up. In particular, the total current flowing into the primary winding at its top end is equal to $i_{p\#1} - i_{p\#2}$. Similarly, the total current flowing into the secondary winding at its top end is equal to $i_{s\#2} - i_{s\#1}$.

Analysis of Case #1 – including only the voltage source in the primary circuit

The principal voltage equation is the voltage drop over the primary resistor R_p:

$$W(t) - R_p\left(i_p + i_{ps}\right) = v_{p\#1}$$

The principal current equation is based on the voltage drop over the parallel branch on the primary side:

$$v_{p\#1} = \frac{L_p}{L_s} R_s i_{ps}$$

The inductor's voltage-current characteristic is:

$$v_{p\#1} = L_p \frac{di_p}{dt}$$

Combining these three equations gives:

$$\left(\frac{L_p}{R_p} + \frac{L_s}{R_s}\right)\frac{di_p}{dt} + i_p = \frac{W(t)}{R_p} \quad (1)$$

We can foresee that the time-constant τ will be equal to:

$$\tau = \frac{L_p}{R_p} + \frac{L_s}{R_s} \quad (2)$$

and that the general solution of Equation (1) will be:

$$i_p = Pe^{-t/\tau} + w(t) \quad (3)$$

where function $w(t)$ will have the same time-dependence as the forcing function $W(t)$. If $W(t)$ is constant, then $w(t)$ will be constant; if $W(t)$ is a sinusoid, then $w(t)$ will be a sinusoid with the same frequency; and so on. The unknown constant P and whatever constants there are in $w(t)$ will be determinable from the initial conditions. Taking the derivative of i_p, and multiplying by L_p, gives $v_{p\#1}$ as:

$$v_{p\#1} = -\frac{PL_p}{\tau} e^{-t/\tau} + L_p \dot{w}(t) \quad (4)$$

where I have used the convention that an overhead dot represents a time-derivative. Let us now turn our attention to the secondary circuit. The voltage $v_{s\#1}$ will be equal to the voltage over the primary circuit $v_{p\#1}$ multiplied by the turns-ratio, thus:

$$v_{s\#1} = \frac{N_s}{N_p} v_{p\#1}$$

Recalling that the inductance-ratio is the square of the turns-ratio, this can be written as:

$$v_{s\#1} = \sqrt{\frac{L_s}{L_p}} v_{p\#1} \quad (5)$$

Then, the current flowing in the secondary circuit can be found from a straight-forward application of Ohm's Law, as:

$$i_{s\#1} = \frac{v_{s\#1}}{R_s} = \frac{1}{R_s}\sqrt{\frac{L_s}{L_p}} v_{p\#1} \quad (6)$$

There are a couple of loose ends to win up. Substituting the expression for $v_{p\#1}$ from Equation (4) into Equations (5) and (6) will give expressions for $v_{s\#1}$ and $i_{s\#1}$ directly in terms of time. Similarly, substituting Equation (4) into the principal current equation will give an expression for i_{ps} directly in terms of time. Finally, adding i_p and i_{ps} together will give $i_{p\#1}$. When these substitutions are done, we have the following equations for the six unknown quantities:

$$i_p = Pe^{-t/\tau} + w(t) \qquad (3)$$

$$v_{p\#1} = -\frac{PL_p}{\tau} e^{-t/\tau} + L_p \dot{w}(t) \qquad (4)$$

$$i_{ps} = \frac{L_s}{R_s}\left[-\frac{P}{\tau} e^{-t/\tau} + \dot{w}(t)\right] \qquad (7)$$

$$i_{p\#1} = Pe^{-t/\tau} + w(t) + \frac{L_s}{R_s}\left[-\frac{P}{\tau} e^{-t/\tau} + \dot{w}(t)\right] \qquad (8)$$

$$v_{s\#1} = \sqrt{L_p L_s}\left[-\frac{P}{\tau} e^{-t/\tau} + \dot{w}(t)\right] \qquad (9)$$

$$i_{s\#1} = \frac{\sqrt{L_p L_s}}{R_s}\left[-\frac{P}{\tau} e^{-t/\tau} + \dot{w}(t)\right] \qquad (10)$$

<u>The relationship between $W(t)$ and $w(t)$</u>

In the above analysis, I was not as precise about $w(t)$ as I could have been. Let us look again at $W(t)$. The differential equation for i_p was given in Equation (1), which is repeated here for convenience:

$$\tau \frac{di_p}{dt} + i_p = \frac{W(t)}{R_p} \quad (1)$$

We proposed a solution of form $i_p = Pe^{-t/\tau} + w(t)$. Substituting i_p and its derivative into the differential equation gives:

$$-\tau \frac{P}{\tau} e^{-t/\tau} + \tau \dot{w}(t) + Pe^{-t/\tau} + w(t) = \frac{W(t)}{R_p}$$

The way we selected the time-constant τ above guarantees satisfaction of the transient part of the solution. That is, the two terms in $e^{-t/\tau}$ cancel each of out, leaving:

$$\tau\dot{w}(t) + w(t) = \frac{W(t)}{R_p} \qquad (11)$$

This differential equation will need to be solved for whatever forcing function $W(t)$ is given. A couple of common cases are the following:

A. If $W(t) = W_0$ is constant, then, by inspection, $w(t) = W_0/R_p$ (12)

B. If $W(t) = W_0 \sin 2\pi f t$, then $w(t)$ will be a sinusoid having the form $w(t) = A \sin 2\pi f t + B \cos 2\pi f t$ with derivative $\dot{w}(t) = 2\pi f A \cos 2\pi f t - 2\pi f B \sin 2\pi f t$. Substituting into Equation (11),

$$2\pi f\tau A \cos 2\pi f t - 2\pi f \tau B \sin 2\pi f t + A \sin 2\pi f t + B \cos 2\pi f t = \frac{W_0}{R_p} \sin 2\pi f t$$

$$\rightarrow \quad (2\pi f \tau A + B) \cos 2\pi f t + \left(-2\pi f \tau B + A - \frac{W_0}{R_p}\right) \sin 2\pi f t = 0$$

Since the cosine and sine functions are independent from each other with respect to time, the only way this equation can hold true for any and all times t is if the coefficient of each term is identically zero. Then,

$$B = -2\pi f \tau A$$
$$A = 2\pi f \tau B + \frac{W_0}{R_p}$$

These two equations can be solved for A and B:

$$A = \frac{W_0}{R_p(1 + 4\pi^2 f^2 \tau^2)}$$
$$B = -\frac{2\pi f \tau W_0}{R_p(1 + 4\pi^2 f^2 \tau^2)}$$

from which we can write down the equation for $w(t)$:

$$w(t) = \frac{W_0}{R_p(1 + 4\pi^2 f^2 \tau^2)} (\sin 2\pi f t - 2\pi f \tau \cos 2\pi f t) \qquad (13)$$

C. If $W(t) = W_0 e^{j2\pi f t}$, then $w(t)$ will be an exponential function with an imaginary exponent having the form $w(t) = De^{j2\pi f t} + E$ with derivative $\dot{w}(t) = j2\pi f D e^{j2\pi f t}$. Substituting into Equation (11), we get:

$$j2\pi f \tau D e^{j2\pi f t} + De^{j2\pi f t} + E = \frac{W_0 e^{j2\pi f t}}{R_p}$$

$$\rightarrow \quad \left(j2\pi f \tau D + D - \frac{W_0}{R_p}\right) e^{j2\pi f t} + E = 0$$

Since the exponential function is not constant (other than the degenerate case of zero frequency), the only way this equation can hold true for any and all times t is if the coefficient of each term is identically zero. Then,

$$D = \frac{W_0}{R_p(1 + j2\pi f\tau)}$$
$$E = 0$$

and the expression for $w(t)$ is:

$$w(t) = \frac{W_0}{R_p(1 + j2\pi f\tau)} e^{j2\pi f t} \quad (14)$$

For the moment, I will say nothing about the interpretation of imaginary numbers other than to say that the imaginary coefficient is a more handy way to account for phase angles than the equivalent combination of sinusoidal terms.

Analysis of Case #2 – including only the voltage source in the secondary circuit

The analysis of Case #2 proceeds in the same way as the analysis of Case #1. The time-constant τ will be the same as in Case #1. This time, the differential equation will involve the current i_s (instead of i_p) and it will have a general solution with the form:

$$i_s = Qe^{-t/\tau} + v(t) \quad (15)$$

where function v(t) will have the same time-dependence as the forcing function $V(t)$. The constant Q will have to be found from the initial conditions. We can write down the circuit variables by direct comparison with the results of Case #1, thus:

$$i_s = Qe^{-t/\tau} + v(t) \quad (16)$$
$$v_{s\#2} = -\frac{QL_s}{\tau} e^{-t/\tau} + L_s\dot{v}(t) \quad (17)$$
$$i_{sp} = \frac{L_p}{R_p}\left[-\frac{Q}{\tau} e^{-t/\tau} + \dot{v}(t)\right] \quad (18)$$
$$i_{s\#2} = Qe^{-t/\tau} + v(t) + \frac{L_p}{R_p}\left[-\frac{Q}{\tau} e^{-t/\tau} + \dot{v}(t)\right] \quad (19)$$
$$v_{p\#2} = \sqrt{L_s L_p}\left[-\frac{Q}{\tau} e^{-t/\tau} + \dot{v}(t)\right] \quad (20)$$
$$i_{p\#2} = \frac{\sqrt{L_s L_p}}{R_p}\left[-\frac{Q}{\tau} e^{-t/\tau} + \dot{v}(t)\right] \quad (21)$$

Furthermore, the relationship between $V(t)$ and $v(t)$ will be the same as that between $W(t)$ and $w(t)$, and $v(t)$ must be found by solving the differential equation:

$$\tau\dot{v}(t) + v(t) = \frac{V(t)}{R_s} \quad (22)$$

Adding together the results from Case #1 and Case #2

We now need to combine the results from the two analyses. To do this, we need to refer back to the two schematic models at the bottom of page 1. The voltages in the two cases have the same direction: from top to bottom. But, the currents have been drawn in opposite directions: flowing out of the top of the inductor in the power-on side and into the top of the inductor in the power-off side. Let us define the total currents, $i_p\big|_{total}$ and $i_s|_{total}$, to be positive when flowing into the top of their respective inductors. Then:

$$\begin{aligned} i_p\big|_{total} &= i_{p\#1} - i_{p\#2} \\ v_p\big|_{total} &= v_{p\#1} + v_{p\#2} \\ i_s|_{total} &= i_{s\#2} - i_{s\#1} \\ v_s|_{total} &= v_{s\#1} + v_{s\#2} \end{aligned}$$

Inserting the results for the two cases above, we get:

$$\begin{aligned} i_p\big|_{total} &= Pe^{-t/\tau} + w(t) + \frac{L_s}{R_s}\left[-\frac{P}{\tau}e^{-t/\tau} + \dot{w}(t)\right] - \frac{\sqrt{L_pL_s}}{R_p}\left[-\frac{Q}{\tau}e^{-t/\tau} + \dot{v}(t)\right] \\ v_p\big|_{total} &= -\frac{PL_p}{\tau}e^{-t/\tau} + L_p\dot{w}(t) + \sqrt{L_pL_s}\left[-\frac{Q}{\tau}e^{-t/\tau} + \dot{v}(t)\right] \\ i_s|_{total} &= Qe^{-t/\tau} + v(t) + \frac{L_p}{R_p}\left[-\frac{Q}{\tau}e^{-t/\tau} + \dot{v}(t)\right] - \frac{\sqrt{L_pL_s}}{R_s}\left[-\frac{P}{\tau}e^{-t/\tau} + \dot{w}(t)\right] \\ v_s|_{total} &= \sqrt{L_pL_s}\left[-\frac{P}{\tau}e^{-t/\tau} + \dot{w}(t)\right] - \frac{QL_s}{\tau}e^{-t/\tau} + L_s\dot{v}(t) \end{aligned}$$

Terms can be collected to give:

$$\begin{aligned} i_p\big|_{total} &= \frac{1}{\tau}\left(\tau P - \frac{L_s}{R_s}P + \frac{\sqrt{L_pL_s}}{R_p}Q\right)e^{-t/\tau} + w(t) + \frac{L_s}{R_s}\dot{w}(t) - \frac{\sqrt{L_pL_s}}{R_p}\dot{v}(t) \\ v_p\big|_{total} &= -\frac{1}{\tau}\left(PL_p + \sqrt{L_pL_s}Q\right)e^{-t/\tau} + L_p\dot{w}(t) + \sqrt{L_pL_s}\dot{v}(t) \\ i_s|_{total} &= \frac{1}{\tau}\left(\tau Q - \frac{L_p}{R_p}Q + \frac{\sqrt{L_pL_s}}{R_s}P\right)e^{-t/\tau} + v(t) + \frac{L_p}{R_p}\dot{v}(t) - \frac{\sqrt{L_pL_s}}{R_s}\dot{w}(t) \\ v_s|_{total} &= -\frac{1}{\tau}\left(QL_s + \sqrt{L_pL_s}P\right)e^{-t/\tau} + L_s\dot{v}(t) + \sqrt{L_pL_s}\dot{w}(t) \end{aligned}$$

A little further reduction can be made by substituting the time-constant τ from Equation (2):

$$\left.\begin{aligned} i_p\big|_{total} &= \frac{\sqrt{L_p}}{\tau R_p}\left(\sqrt{L_p}P + \sqrt{L_s}Q\right)e^{-t/\tau} + w(t) + \frac{L_s}{R_s}\dot{w}(t) - \frac{\sqrt{L_pL_s}}{R_p}\dot{v}(t) \\ v_p\big|_{total} &= -\frac{\sqrt{L_p}}{\tau}\left(\sqrt{L_p}P + \sqrt{L_s}Q\right)e^{-t/\tau} + L_p\dot{w}(t) + \sqrt{L_pL_s}\dot{v}(t) \end{aligned}\right\} \quad (23 - primary)$$

$$\left.\begin{aligned} i_s\big|_{total} &= \frac{\sqrt{L_s}}{\tau R_s}\left(\sqrt{L_p}P + \sqrt{L_s}Q\right)e^{-t/\tau} + v(t) + \frac{L_p}{R_p}\dot{v}(t) - \frac{\sqrt{L_p L_s}}{R_s}\dot{w}(t) \\ v_s\big|_{total} &= -\frac{\sqrt{L_s}}{\tau}\left(\sqrt{L_p}P + \sqrt{L_s}Q\right)e^{-t/\tau} + L_s\dot{v}(t) + \sqrt{L_p L_s}\dot{w}(t) \end{aligned}\right\} \quad (23 - secondary)$$

Note the perfect symmetry between the primary winding and the secondary winding. This was to be expected.

<u>Numerical example #1 – Two constant voltages</u>

Let us look at a special case, in which both voltage sources are constant dc. While this is not a common case, it does allow us to focus on the transient response only, which will not be overwhelmed by changes due to the driving voltages. We will use the symbols W_0 and V_0 for the two constant voltage sources, in the primary and secondary circuits, respectively. Then:

$$\begin{aligned} W(t) &= W_0 \qquad & V(t) &= V_0 \\ w(t) &= \frac{W_0}{R_p} \qquad & v(t) &= \frac{V_0}{R_s} \\ \dot{w}(t) &= 0 \qquad & \dot{v}(t) &= 0 \end{aligned}$$

For convenience, let us also assume that power is applied at time $t = 0$. Since the circuit starts up from rest, the magnetizing currents i_p and i_s must be zero at time $t = 0$. From Equations (3) and (15), we can see that:

$$P = -w(0) = -\frac{W_0}{R_p} \qquad Q = -v(0) = -\frac{V_0}{R_s}$$

The set of Equations (23) then reduces to:

$$\left.\begin{aligned} i_p\big|_{total} &= -\frac{\sqrt{L_p}}{\tau R_p}\left(\sqrt{L_p}\frac{W_0}{R_p} + \sqrt{L_s}\frac{V_0}{R_s}\right)e^{-t/\tau} + \frac{W_0}{R_p} \\ v_p\big|_{total} &= \frac{\sqrt{L_p}}{\tau}\left(\sqrt{L_p}\frac{W_0}{R_p} + \sqrt{L_s}\frac{V_0}{R_s}\right)e^{-t/\tau} \\ i_s\big|_{total} &= -\frac{\sqrt{L_s}}{\tau R_s}\left(\sqrt{L_p}\frac{W_0}{R_p} + \sqrt{L_s}\frac{V_0}{R_s}\right)e^{-t/\tau} + \frac{V_0}{R_s} \\ v_s\big|_{total} &= \frac{\sqrt{L_s}}{\tau}\left(\sqrt{L_p}\frac{W_0}{R_p} + \sqrt{L_s}\frac{V_0}{R_s}\right)e^{-t/\tau} \end{aligned}\right\} \quad (24)$$

Notice that the factors in the round brackets are all the same. The start-up currents and voltages, at time $t = 0$, are equal to:

$$\left.\begin{aligned}
i_p\big|_{total}(t=0) &= -\frac{\sqrt{L_p}}{\tau R_p}\left(\sqrt{L_p}\frac{W_0}{R_p}+\sqrt{L_s}\frac{V_0}{R_s}\right)+\frac{W_0}{R_p} \\
v_p\big|_{total}(t=0) &= \frac{\sqrt{L_p}}{\tau}\left(\sqrt{L_p}\frac{W_0}{R_p}+\sqrt{L_s}\frac{V_0}{R_s}\right) \\
i_s|_{total}(t=0) &= -\frac{\sqrt{L_s}}{\tau R_s}\left(\sqrt{L_p}\frac{W_0}{R_p}+\sqrt{L_s}\frac{V_0}{R_s}\right)+\frac{V_0}{R_s} \\
v_s|_{total}(t=0) &= \frac{\sqrt{L_s}}{\tau}\left(\sqrt{L_p}\frac{W_0}{R_p}+\sqrt{L_s}\frac{V_0}{R_s}\right)
\end{aligned}\right\} \qquad (25)$$

We can see that the ratio of the initial voltages is equal to the turns-ratio, which is itself equal to the square root of the inductance ratio, as it must:

$$\frac{v_p\big|_{total}(t=0)}{v_s|_{total}(t=0)} = \frac{\sqrt{L_p}}{\sqrt{L_s}} = \frac{N_p}{N_s}$$

In order to examine the ratio of the initial currents, we need to collect terms first. The expression for $i_p\big|_{total}(t=0)$ is:

$$\begin{aligned}
i_p\big|_{total}(t=0) \quad &= -\frac{L_p W_0}{\tau R_p^2} - \frac{\sqrt{L_p L_s} V_0}{\tau R_p R_s} + \frac{W_0}{R_p} \\
&= \frac{W_0}{R_p}\left(\frac{\tau R_p - L_p}{\tau R_p}\right) - \frac{\sqrt{L_p L_s} V_0}{\tau R_p R_s} \\
&= \frac{W_0}{R_p}\left(\frac{L_p + \frac{R_p L_s}{R_s} - L_p}{\tau R_p}\right) - \frac{\sqrt{L_p L_s} V_0}{\tau R_p R_s} \\
&= \frac{L_s W_0 - \sqrt{L_p L_s} V_0}{\tau R_p R_s}
\end{aligned}$$

Similarly, the expression for $i_s|_{total}(t=0)$ can be written as:

$$\begin{aligned}
i_s|_{total}(t=0) \quad &= -\frac{\sqrt{L_p L_s} W_0}{\tau R_p R_s} - \frac{L_s V_0}{\tau R_s^2} + \frac{V_0}{R_s} \\
&= \frac{V_0}{R_s}\left(\frac{\tau R_s - L_s}{\tau R_s}\right) - \frac{\sqrt{L_p L_s} W_0}{\tau R_p R_s} \\
&= \frac{V_0}{R_s}\left(\frac{\frac{L_p R_s}{R_p} + L_s - L_s}{\tau R_s}\right) - \frac{\sqrt{L_p L_s} W_0}{\tau R_p R_s} \\
&= \frac{L_p V_0 - \sqrt{L_p L_s} W_0}{\tau R_p R_s}
\end{aligned}$$

Then, the ratio of the initial currents is equal to:

$$\begin{aligned}\frac{i_p\big|_{total}(t=0)}{i_s\big|_{total}(t=0)} &= \frac{L_sW_0-\sqrt{L_pL_s}V_0}{L_pV_0-\sqrt{L_pL_s}W_0}\\ &= \frac{\sqrt{L_s}}{\sqrt{L_p}}\left(\frac{\sqrt{L_s}W_0-\sqrt{L_p}V_0}{\sqrt{L_p}V_0-\sqrt{L_s}W_0}\right)\\ &= -\frac{\sqrt{L_s}}{\sqrt{L_p}} = -\frac{N_s}{N_p}\end{aligned}$$

This is correct, including the minus sign. The minus sign occurs because we defined both currents as being positive when flowing into the tops of their respective inductors. This is the reverse of the customary labeling, in which the current from the secondary flows out of the top of its inductor. The magnitude is also correct, being the reciprocal of the turns-ratio.

As time passes, the transient dies away. For large enough times t, often quantified as being more than five time-constants, the currents and voltages steady out at the following values:

$$\begin{aligned} i_p\big|_{total}(t\gg\tau) &= \frac{W_0}{R_p}\\ v_p\big|_{total}(t\gg\tau) &= 0\\ i_s\big|_{total}(t\gg\tau) &= \frac{V_0}{R_s}\\ v_s\big|_{total}(t\gg\tau) &= 0\end{aligned}$$

Let us look at the following numerical values:

$$\begin{array}{cc} W_0 = 10\text{V} & V_0 = 6\text{V}\\ L_p = 50\text{mH} & L_s = 100\text{mH}\\ R_p = 10\Omega & R_s = 25\Omega\end{array}$$

The time-constant (there is only one time-constant), is equal to:

$$\tau = \frac{L_p}{R_p}+\frac{L_s}{R_s} = \frac{0.05}{10}+\frac{0.1}{25} = 0.009\text{s}$$

The primary-side inductor-resistor pair has an individual time-constant of 5ms. The secondary-side inductor-resistor pair has an individual time-constant of 4ms. The individual time-constants combine is such a way that the time-constant of the circuit as a whole is their sum. The steady-state voltages and currents are:

$$\begin{aligned} i_p\big|_{total}(t\gg\tau) &= \frac{W_0}{R_p} = \frac{10}{10} = 1\text{A}\\ v_p\big|_{total}(t\gg\tau) &= 0\\ i_s\big|_{total}(t\gg\tau) &= \frac{V_0}{R_s} = \frac{6}{25} = 240\text{mA}\\ v_s\big|_{total}(t\gg\tau) &= 0\end{aligned}$$

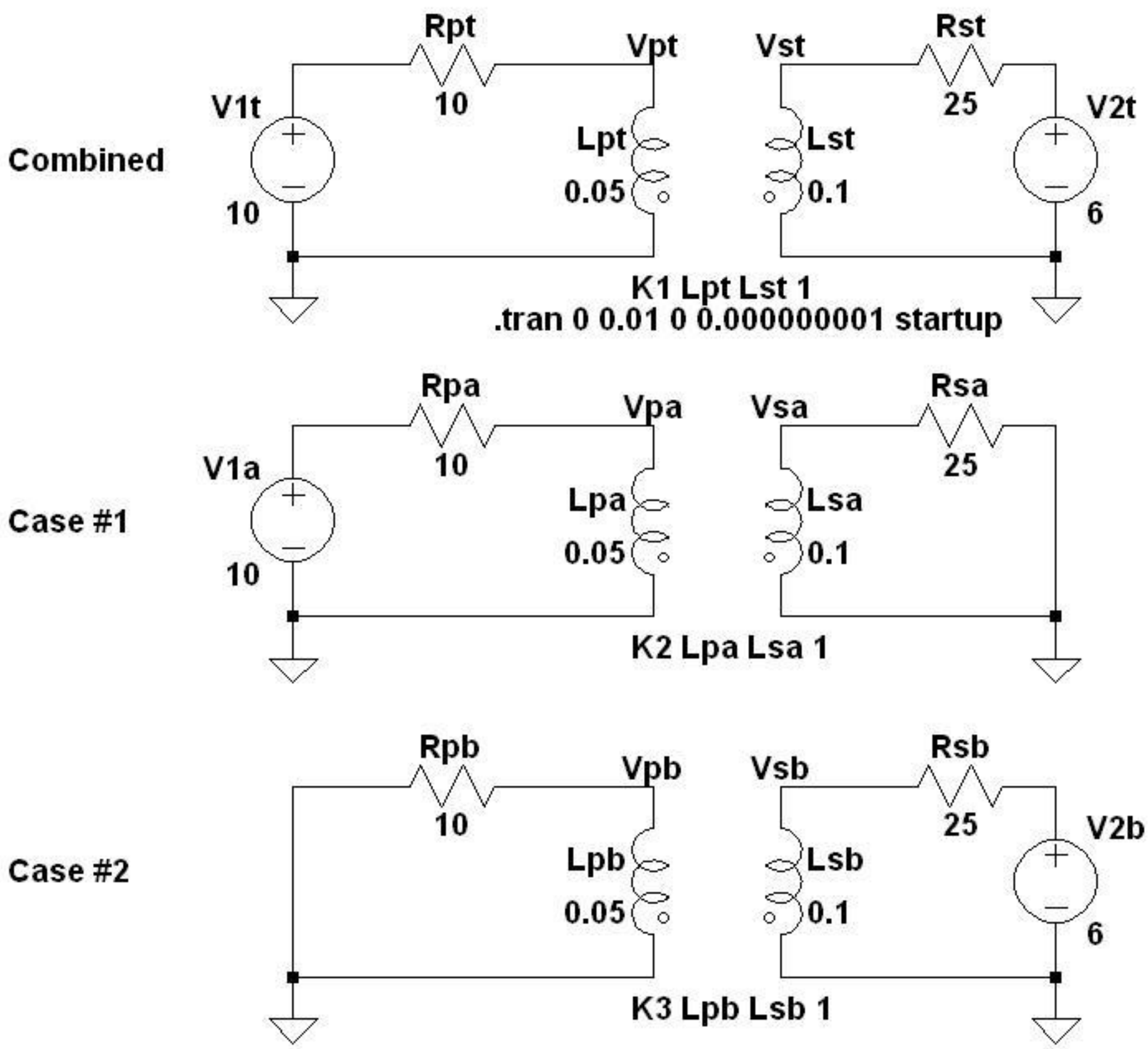
The factor in the round brackets in Equation (25) is equal to:

$$\sqrt{L_p}\frac{W_0}{R_p}+\sqrt{L_s}\frac{V_0}{R_s}=\sqrt{0.05}\frac{10}{10}+\sqrt{0.1}\frac{6}{25}=0.299501\mathrm{A}\sqrt{\mathrm{H}}$$

so the initial currents and voltages are equal to:

$$i_p\big|_{total}(t=0)=-0.299501\frac{\sqrt{L_p}}{\tau R_p}+\frac{W_0}{R_p}=-\frac{0.299501}{0.009}\frac{\sqrt{0.05}}{10}+\frac{10}{10}=255.9\mathrm{mA}$$
$$v_p\big|_{total}(t=0)=0.299501\frac{\sqrt{L_p}}{\tau}=\frac{0.299501}{0.009}\sqrt{0.05}=7.441\mathrm{V}$$
$$i_s|_{total}(t=0)=-0.299501\frac{\sqrt{L_s}}{\tau R_s}+\frac{V_0}{R_s}=-\frac{0.299501}{0.009}\frac{\sqrt{0.1}}{25}+\frac{6}{25}=-180.9\mathrm{mA}$$
$$v_s|_{total}(t=0)=0.299501\frac{\sqrt{L_s}}{\tau}=\frac{0.299501}{0.009}\sqrt{0.1}=10.52\mathrm{V}$$

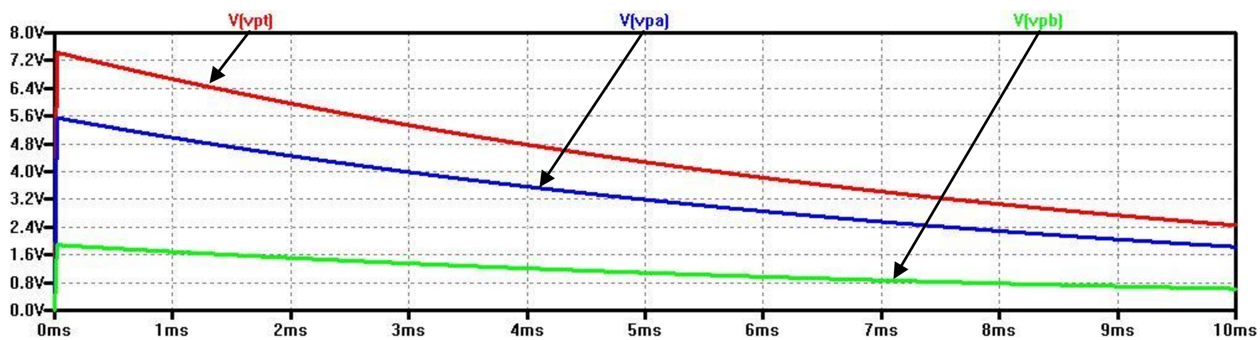
I carried out a spice simulation of this circuit using the following SPICE schematic.



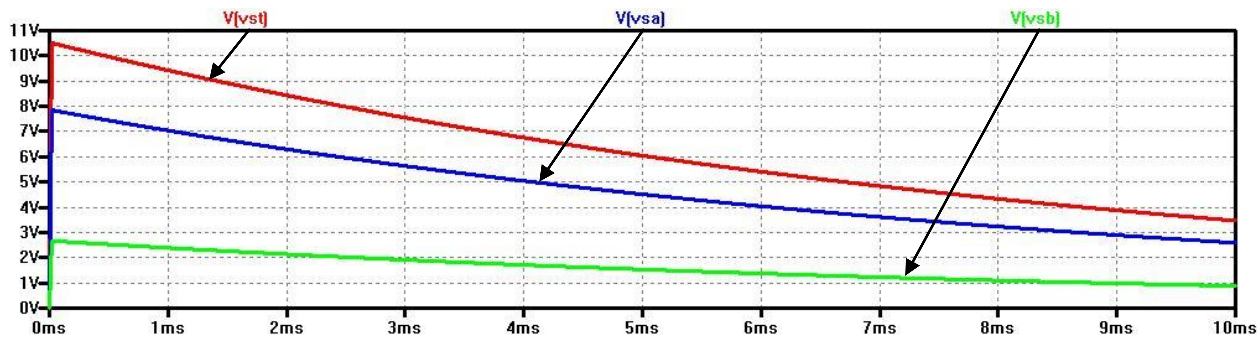
The schematic includes three independent, but very similar, circuits. The middle circuit models Case #1 (with the primary-side voltage source only) and the bottom circuit models Case #2 (with the secondary-side voltage source only). The combined circuit (with both voltage sources) is at the top. All component values and voltage tags have two subscripts. One set of subscripts indicates "p" for primary-side or "s" for secondary-side. The second set of subscripts indicates "t" for the total, or combined, circuit, "a" for the Case #1 circuit and "b" for the Case #2 circuit. When the simulation is run, all three circuits are integrated independently. Afterwards, various waveforms can be plotted and the relationships between the two cases and the combined circuit examined.

The mutual inductance directive, of form K1 Lpt Lst 1, specifies that the linkage is 100% – that the transformer is ideal. The resistors shown are assumed to include the series resistance of the windings in their circuits. Note that the current convention used by SPICE is that current flowing through an inductor is negative if it flows into the dotted end and positive if it flows into the undotted end.

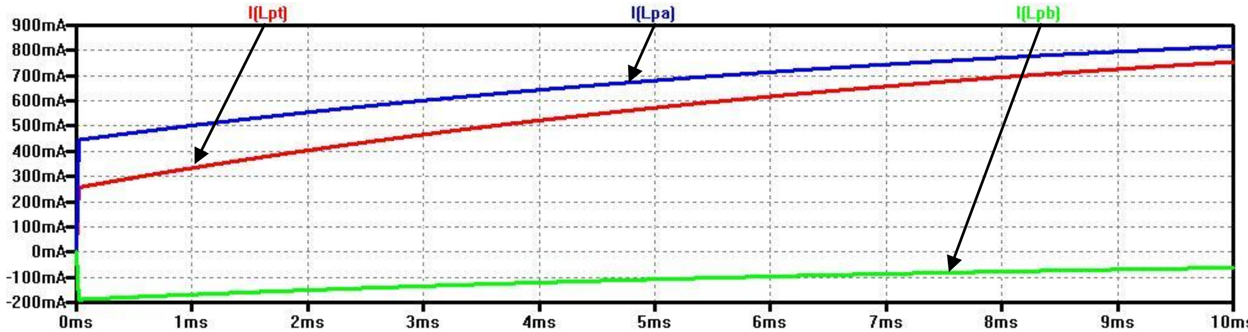
Let us look first at the voltages on the primary side. The combined voltage ($v_p\big|_{total}$ in the analysis above and V(vpt) in the graph) starts at 7.441V and decays to zero with a time-constant of 9ms.



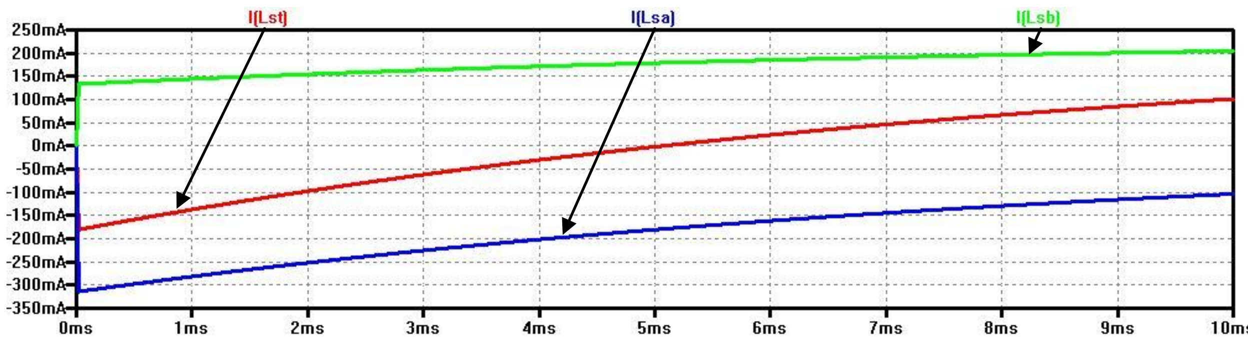
The voltages on the secondary side are shown in the following graph. The combined voltage ($v_s|_{total}$ in the analysis above and V(vst) in the graph) starts at 10.52V and decays to zero with a time-constant of 9ms.



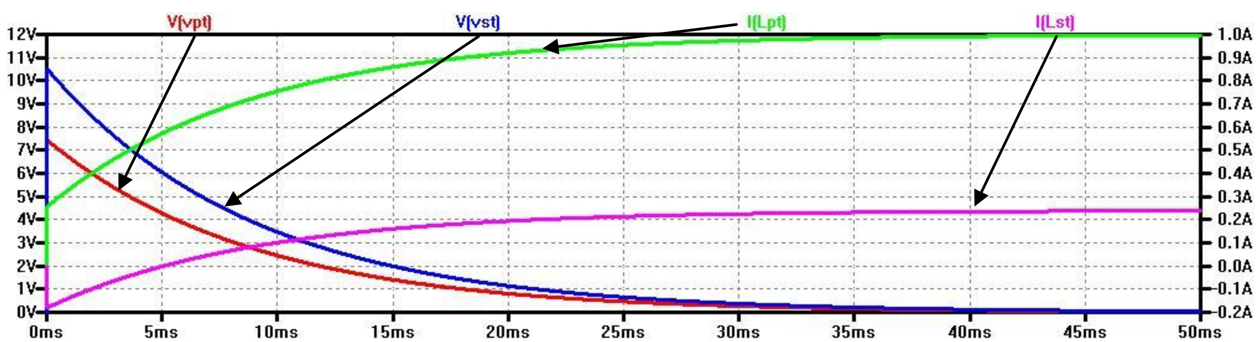
The currents on the primary side are shown in the following graph. The combined current ($i_p\big|_{total}$ in the analysis above and I(Lpt) in the graph) starts at 255.9mA and rises to 1A with a time-constant of 9ms.



The currents on the secondary side are shown in the following graph. The combined current ($i_s|_{total}$ in the analysis above and I(Lst) in the graph) starts at negative 180.9mA and rises to positive 240mA with a time-constant of 9ms. This steady-state current is not so well illustrated in the graph, which shows only a single time-constant's length of time. Running the simulation out to 50ms confirms that I(Lst) does rise all the way to 240mA.



In fact, all variables converge to their expected steady-state values. This is confirmed by the following graph which shows the waveforms for 50ms, slightly longer than five time-constants.



The careful reader will observe that all the waveforms exhibit a spike at time $t = 0$. This is an artifact of the SPICE integration, which assumes that constant voltage sources do not start up instantaneously. Instead, SPICE assumes that constant voltage sources power up over 20µs, during which time their voltages rise linearly from zero to their constant values.

Numerical example #2 – Two sinusoidal driving voltages

Having looked at the transient response, let us now look at a second example, in which we will focus on the steady-state. We will look at the case where both voltage sources are sinusoidal. In particular, we will assume that the driving voltages have the mathematical forms:

$$W(t) = W_0 e^{2\pi f t} \qquad V(t) = V_0 e^{2\pi g t}$$

We will allow the voltage sources to have different temporal frequencies, f and g, as well as different amplitudes W_0 and V_0. For convenience, though, they will start at zero. An incidental benefit from this choice is that there is no transient voltage response at all. The circuit is in its steady-state mode right from the get-go.

We will have to solve differential Equations (11) and (22) to find $w(t)$ and $v(t)$. However, we have already done so above, in the sub-case C, and can take over the result from Equation (14) to write:

$$w(t) = \frac{W_0}{R_p(1 + j2\pi f\tau)} e^{j2\pi f t}$$
$$v(t) = \frac{V_0}{R_s(1 + j2\pi g\tau)} e^{j2\pi g t}$$

We will need the derivatives as well:

$$\dot{w}(t) = \frac{j2\pi f W_0}{R_p(1 + j2\pi f\tau)} e^{j2\pi f t}$$
$$\dot{v}(t) = \frac{j2\pi g V_0}{R_s(1 + j2\pi g\tau)} e^{j2\pi g t}$$

We can now substitute these waveforms into the general Equations (23). For times much greater than the time-constant (and, in this case, for shorter times, too), the steady-state parts of Equations (23) are:

$$\left.\begin{aligned} i_p\big|_{total}(t \gg \tau) &= w(t) + \frac{L_s}{R_s}\dot{w}(t) - \frac{\sqrt{L_p L_s}}{R_p}\dot{v}(t) \\ v_p\big|_{total}(t \gg \tau) &= L_p\dot{w}(t) + \sqrt{L_p L_s}\,\dot{v}(t) \\ i_s\big|_{total}(t \gg \tau) &= v(t) + \frac{L_p}{R_p}\dot{v}(t) - \frac{\sqrt{L_p L_s}}{R_s}\dot{w}(t) \\ v_s\big|_{total}(t \gg \tau) &= L_s\dot{v}(t) + \sqrt{L_p L_s}\,\dot{w}(t) \end{aligned}\right\} \qquad (26)$$

Substituting the sinusoidal driving functions gives:

$$\left.\begin{aligned}
i_p\big|_{total}(t \gg \tau) &= \frac{W_0}{R_p(1+j2\pi f\tau)}\left(1+j2\pi f\frac{L_s}{R_s}\right)e^{j2\pi ft} - \frac{\sqrt{L_pL_s}}{R_pR_s}\frac{j2\pi gV_0}{(1+j2\pi g\tau)}e^{j2\pi gt} \\
v_p\big|_{total}(t \gg \tau) &= L_p\frac{j2\pi fW_0}{R_p(1+j2\pi f\tau)}e^{j2\pi ft} + \sqrt{L_pL_s}\frac{j2\pi gV_0}{R_s(1+j2\pi g\tau)}e^{j2\pi gt} \\
i_s\big|_{total}(t \gg \tau) &= \frac{V_0}{R_s(1+j2\pi g\tau)}\left(1+j2\pi g\frac{L_p}{R_p}\right)e^{j2\pi gt} - \frac{\sqrt{L_pL_s}}{R_sR_p}\frac{j2\pi fW_0}{(1+j2\pi f\tau)}e^{j2\pi ft} \\
v_s\big|_{total}(t \gg \tau) &= L_s\frac{j2\pi gV_0}{R_s(1+j2\pi g\tau)}e^{j2\pi gt} + \sqrt{L_pL_s}\frac{j2\pi fW_0}{R_p(1+j2\pi f\tau)}e^{j2\pi ft}
\end{aligned}\right\} \quad (27)$$

Each of the four independent variables is the sum of two terms, one being a sinusoid at temporal frequency f and the other being a sinusoid at frequency g. Each sinusoidal term is multiplied by a factor which is a constant imaginary number. Each such factor can be expanded into the standard imaginary form $P + jQ$, where both P and Q are strictly real numbers. Then, each such factor can in turn be expressed in the standard exponential form Re^{jS}, where both R and S are strictly real numbers. Since there are eight terms, there will be eight amplitudes, which we will call R_i for $i = 1, 2, 3 \cdots 8$, and eight phase angles, which we will call S_i for $i = 1, 2, 3 \cdots 8$. With these symbols, Equations (27) can be written as:

$$\left.\begin{aligned}
i_p\big|_{total}(t \gg \tau) &= R_1e^{j(2\pi ft+S_1)} + R_2e^{j(2\pi gt+S_2)} \\
v_p\big|_{total}(t \gg \tau) &= R_3e^{j(2\pi ft+S_3)} + R_4e^{j(2\pi gt+S_4)} \\
i_s\big|_{total}(t \gg \tau) &= R_5e^{j(2\pi ft+S_5)} + R_6e^{j(2\pi gt+S_6)} \\
v_s\big|_{total}(t \gg \tau) &= R_7e^{j(2\pi ft+S_7)} + R_8e^{j(2\pi gt+S_8)}
\end{aligned}\right\} \quad (28)$$

Calculating the R_is and S_is is tedious, but not difficult. We will do one pair, for R_1 and S_1, to show the procedure.

$$\begin{aligned}
P_1 + jQ_1 \quad &= \frac{W_0}{R_p(1+j2\pi f\tau)}\left(1+j2\pi f\frac{L_s}{R_s}\right) \\
&= \frac{W_0}{R_p(1+j2\pi f\tau)}\frac{(1-j2\pi f\tau)}{(1-j2\pi f\tau)}\left(1+j2\pi f\frac{L_s}{R_s}\right) \\
&= \frac{W_0\left(1-j2\pi f\tau+j2\pi f\frac{L_s}{R_s}+4\pi^2f^2\tau\frac{L_s}{R_s}\right)}{R_p(1+4\pi^2f^2\tau^2)} \\
&= \frac{W_0\left(1+4\pi^2f^2\tau\frac{L_s}{R_s}\right)}{R_p(1+4\pi^2f^2\tau^2)} + j\frac{2\pi fW_0\left(-\tau+\frac{L_s}{R_s}\right)}{R_p(1+4\pi^2f^2\tau^2)} \\
&= \frac{W_0\left(1+4\pi^2f^2\tau\frac{L_s}{R_s}\right)}{R_p(1+4\pi^2f^2\tau^2)} + j\frac{-2\pi fW_0\left(\frac{L_p}{R_p}\right)}{R_p(1+4\pi^2f^2\tau^2)}
\end{aligned}$$

Then, the standard exponential form is found by:

$$
\begin{aligned}
R_1 e^{jS_1} &= \sqrt{P_1^2 + Q_1^2}\, e^{j\tan^{-1}\left(\frac{Q_1}{P_1}\right)} \\
&= \sqrt{\left[\frac{W_0\left(1 + 4\pi^2 f^2 \tau \frac{L_s}{R_s}\right)}{R_p(1 + 4\pi^2 f^2 \tau^2)}\right]^2 + \left[\frac{-2\pi f W_0\left(\frac{L_p}{R_p}\right)}{R_p(1 + 4\pi^2 f^2 \tau^2)}\right]^2}\; e^{j\tan^{-1}\left[\frac{-2\pi f W_0\left(\frac{L_p}{R_p}\right)}{W_0\left(1+4\pi^2 f^2 \tau \frac{L_s}{R_s}\right)}\right]} \\
&= \frac{W_0}{R_p(1 + 4\pi^2 f^2 \tau^2)} \sqrt{\left(1 + 4\pi^2 f^2 \tau \frac{L_s}{R_s}\right)^2 + 4\pi^2 f^2 \left(\frac{L_p}{R_p}\right)^2}\; e^{j\tan^{-1}\left[\frac{-2\pi f\left(\frac{L_p}{R_p}\right)}{1+4\pi^2 f^2 \tau \frac{L_s}{R_s}}\right]}
\end{aligned}
$$

Regretfully, no further simplification is possible. This is particularly unfortunate because there are seven more pairs like this one.

However, simplification is possible in certain cases where we can make approximations. For example, let us assume that both voltage sources run at "suitably high" frequencies. Let us define "suitably high" using the mathematical restrictions that:

$$2\pi f\tau \gg 1 \qquad\qquad 2\pi g\tau \gg 1$$

For example, if the time-constant is 9ms, as it was in Numerical example #1, then a frequency of 5KHz would satisfy these inequalities because $2\pi f\tau = 2\pi \times 5000 \times 0.009 = 283 \gg 1$. So, if the frequencies f and g satisfy these inequalities, then Equations (27) can themselves be approximated by the following set of equations:

$$
\left.
\begin{aligned}
i_p\big|_{total}(t \gg \tau) &= \frac{W_0 L_s}{\tau R_p R_s} e^{j2\pi f t} - \frac{V_0\sqrt{L_p L_s}}{\tau R_p R_s} e^{j2\pi g t} \\
v_p\big|_{total}(t \gg \tau) &= \frac{W_0 L_p}{\tau R_p} e^{j2\pi f t} + \frac{V_0\sqrt{L_p L_s}}{\tau R_s} e^{j2\pi g t} \\
i_s\big|_{total}(t \gg \tau) &= \frac{V_0 L_p}{\tau R_p R_s} e^{j2\pi g t} - \frac{W_0\sqrt{L_p L_s}}{\tau R_p R_s} e^{j2\pi f t} \\
v_s\big|_{total}(t \gg \tau) &= \frac{V_0 L_s}{\tau R_s} e^{j2\pi g t} + \frac{W_0\sqrt{L_p L_s}}{\tau R_p} e^{j2\pi f t}
\end{aligned}
\right\} \qquad (29)
$$

These can be re-written as:

$$
\left.
\begin{aligned}
i_p\big|_{total}(t \gg \tau) &= \frac{\sqrt{L_s}}{\tau R_p R_s}\left(W_0\sqrt{L_s} e^{j2\pi f t} - V_0\sqrt{L_p} e^{j2\pi g t}\right) \\
v_p\big|_{total}(t \gg \tau) &= \frac{\sqrt{L_p}}{\tau R_p R_s}\left(W_0 R_s\sqrt{L_p} e^{j2\pi f t} + V_0 R_p\sqrt{L_s} e^{j2\pi g t}\right) \\
i_s\big|_{total}(t \gg \tau) &= \frac{\sqrt{L_p}}{\tau R_p R_s}\left(-W_0\sqrt{L_s} e^{j2\pi f t} + V_0\sqrt{L_p} e^{j2\pi g t}\right) \\
v_s\big|_{total}(t \gg \tau) &= \frac{\sqrt{L_s}}{\tau R_p R_s}\left(W_0 R_s\sqrt{L_p} e^{j2\pi f t} + V_0 R_p\sqrt{L_s} e^{j2\pi g t}\right)
\end{aligned}
\right\} \qquad (30)
$$

And, finally, observing the symmetries between the secondary and the primary sides, we have, for the secondary dependent variables:

$$\left.\begin{aligned} i_s\big|_{total}(t \gg \tau) &= -\sqrt{\frac{L_p}{L_s}}\, i_p\big|_{total}(t \gg \tau) \\ v_s\big|_{total}(t \gg \tau) &= \sqrt{\frac{L_s}{L_p}}\, v_p\big|_{total}(t \gg \tau) \end{aligned}\right\} \qquad (31)$$

So, for sinusoids at high frequencies, the secondary current and voltage have the "normal" turns-ratio relationships with their primary circuit equivalents.

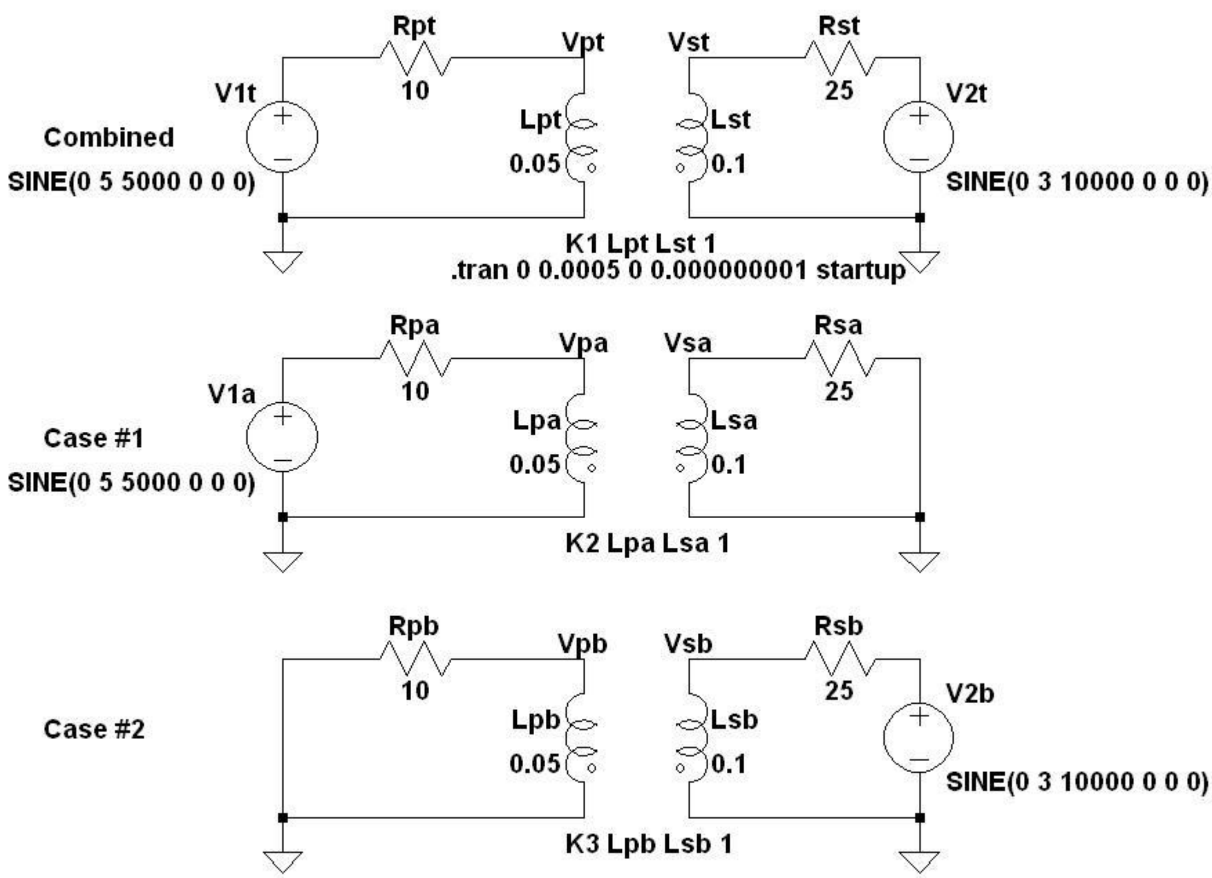
Let us use the same component values as we did in Numerical example #1, namely:

$$L_p = 50\text{mH} \qquad L_s = 100\text{mH}$$
$$R_p = 10\Omega \qquad R_s = 25\Omega$$

The time-constant will therefore be the same as before, 9ms. Let us assume that the voltage sources are:

$$W(t) = 6\sin(2\pi 5000 t) \qquad V(t) = 3\sin(2\pi 10000 t)$$

The following figure is the SPICE model for this analysis. All that has changed from Numerical example #1 is the specification of the voltage sources.



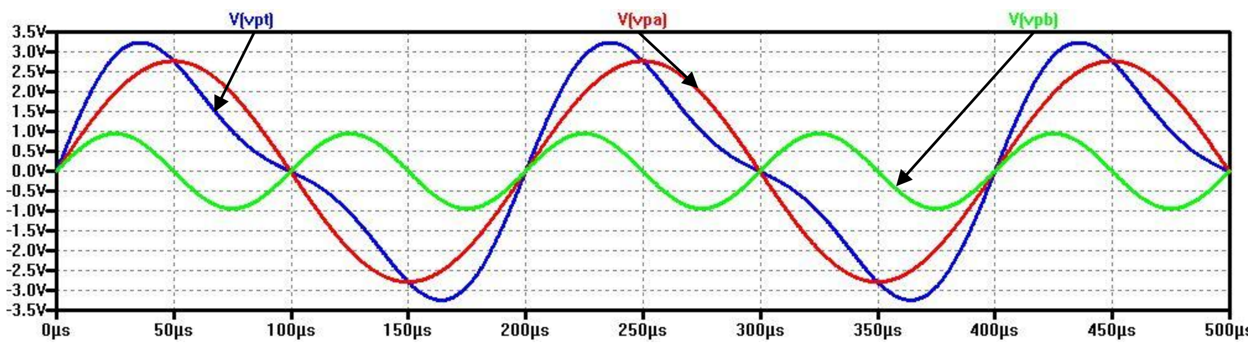
Let us use the results of the analysis to calculate the expected steady-state primary current:

$$
\begin{aligned}
i_p\big|_{total}(t \gg \tau) &= \frac{\sqrt{L_s}}{\tau R_p R_s}\left(W_0\sqrt{L_s}e^{j2\pi f t} - V_0\sqrt{L_p}e^{j2\pi g t}\right) \\
&= \frac{\sqrt{0.1}}{0.009 \times 10 \times 25}\left(6\sqrt{0.1}e^{j2\pi f t} - 3\sqrt{0.05}e^{j2\pi g t}\right) \\
&= 226.7e^{j2\pi f t} - 94.28e^{j2\pi g t}\,\text{mA}
\end{aligned}
$$

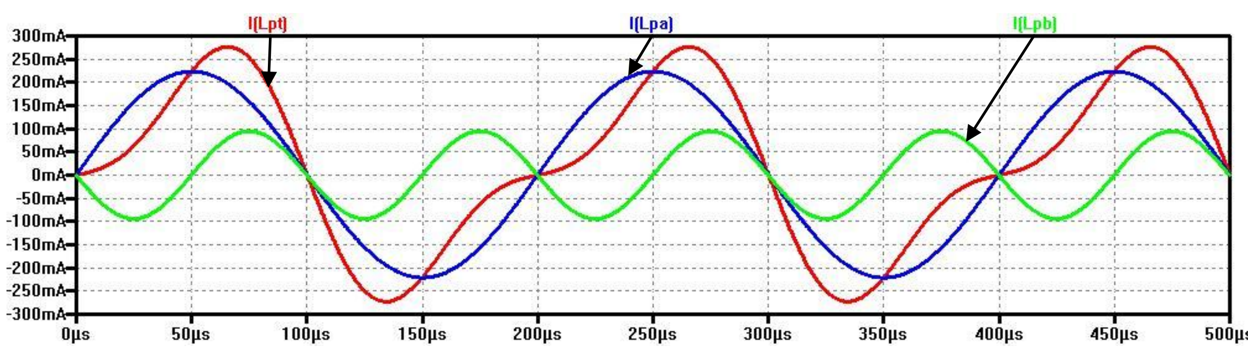
And, for the primary voltage:

$$
\begin{aligned}
v_p\big|_{total}(t \gg \tau) &= \frac{\sqrt{L_p}}{\tau R_p R_s}\left(W_0 R_s\sqrt{L_p}e^{j2\pi f t} + V_0 R_p\sqrt{L_s}e^{j2\pi g t}\right) \\
&= \frac{\sqrt{0.05}}{0.009 \times 10 \times 25}\left(6 \times 25\sqrt{0.05}e^{j2\pi f t} + 3 \times 10\sqrt{0.1}e^{j2\pi g t}\right) \\
&= 3.333e^{j2\pi f t} + 0.9428e^{j2\pi g t}\,\text{V}
\end{aligned}
$$

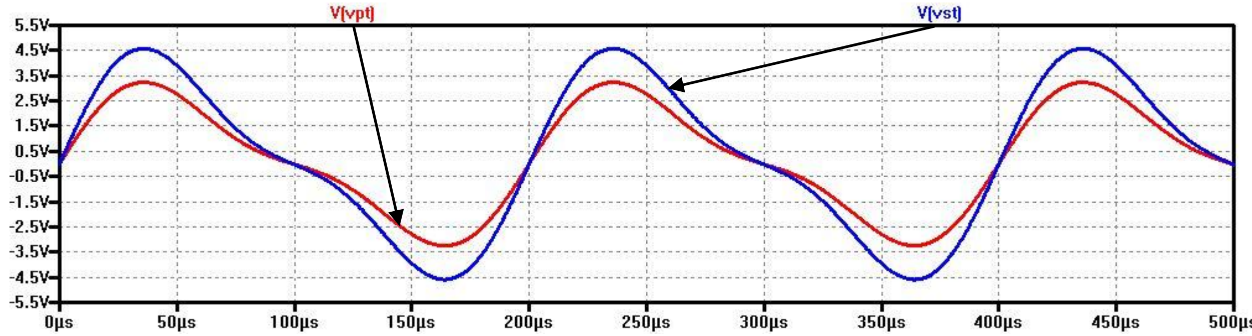
The following chart showing the components of the primary voltage, and how the 3.333V-amplitude contribution from Case #1 and the 0.9428V-amplitude contribution from Case #2 add up to the total voltage in the primary circuit.



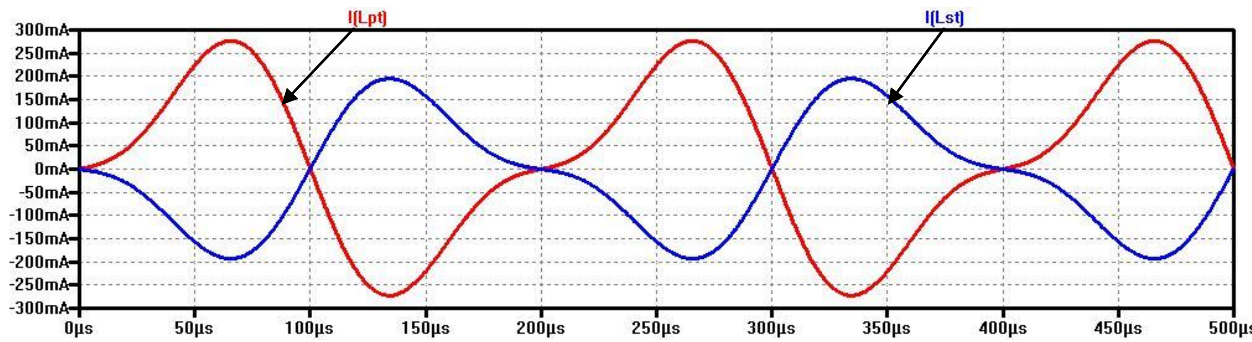
The following chart shows the components of the primary current, and how the 226.7mA-amplitude contribution from Case #1 and the negative 94.28mA-amplitude contribution from Case #2 add up to the total current in the primary circuit.



The following chart shows how the secondary voltage is equal to the primary voltage multiplied by the turns-ratio. The turns-ratio is $\sqrt{0.1/0.05} = 1.414$, so the peak amplitude of the primary voltage (about 3.25V) is scaled up to a peak amplitude of about 4.60V over the secondary winding.



The following chart shows how the secondary current is equal to the primary current multiplied by the reciprocal of the turns-ratio. Since the peak amplitude of the primary current is about 280mA, the peak amplitude of the secondary current will be about 198mA. Recall, too, that the primary and secondary currents have opposing algebraic signs.



One final note: since both voltage sources start up at zero volts, there is no transient response, and the steady-state is entered immediately.

Jim Hawley
December 2011

An e-mail describing errors or omissions would be appreciated.